SCSVMV

Course Material
CLASSICAL ALGEBRA

I BSC MATHEMATICS

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Chapter 1

PARTIAL FRACTIONS

Contents

1.0 AIMS AND OBJECTIVES

Our Aim is to learn about the partial fractions. Further we aim at learning problems to be solved using the different types in partial fractions.

1. 1 PARTIAL FRACTIONS

We are very well acquainted with the process of grouping a number of fractions connected by plus and minus signs into a single fraction. But here in this chapter we shall deal with the process of decomposing a given fraction into a number of fractions connected by plus and minus signs. This process is called splitting into partial fractions and the fractions thus obtained are called the partial fractions of the given fraction,

Example $\dfrac{x^2-10x+13}{x^3-6x^2+11x-6}=\dfrac{2}{x-1}+\dfrac{3}{x-2}+\dfrac{4}{x-3}$.

The fraction on the R.H.S are called the partial Fraction of the fraction on the L.H.S.

1.2 RATIONAL FRACTIONS

Expression of the form $\dfrac{P}{Q}$, where P and Q are polynomials are called **Rational fractions**. Example $\dfrac{x^2+x+1}{x+1}$

1.3 PROPER FRACTIONS

If P is lower degree than Q, $\dfrac{P}{Q}$ is called a **Proper Fraction**s.

Example $\dfrac{x^3+x^2+x+1}{x^4+x+1}$

1.4 IMPROPER FRACTIONS

If P is of degree equal to or greater than that of Q then $\frac{P}{Q}$ is called an Improper fraction.

We can always express an improper fraction as the sum of a polynomial and a proper fraction.

For example $\frac{x^2+x+1}{x+1}$

1. 5 SOME FUNDAMENTAL RULES OF PARTIAL FRACTIONS

1. Before splitting into a fraction into its partial fraction it must be seen that the given fraction is a proper fraction. If it is not, then divide the numerator by denominator and express as ‘Quotient +Proper fraction’ and then the proper fraction thus obtained will be broken into Partial Fractions.

2. The denominator shall be decomposed into as many fractions as possible, e.g., a factor $(1-x^4)$ should be written as $(1-x)(1+x)(1+x^2)$.

3. To every linear factor (non repeated) of the form $(1-a)$ in the denominator there will be a P.F. of the form $\frac{A}{x-a}$, where A is a constant to be determined. Example: $\frac{3x+7}{(x-3)(x-2)(x-1)}$ will have its P.F’s of the form $\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

4. To every linear factor repeated k times of the form $(x-a)^k$ in the denominator, there will be P.F’s of the form $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+........+\frac{A_k}{(ax+b)^k}$

Example:
$\frac{x^2+1}{(2x-3)(x-1)^2}$ will have its P.F’s of the form $\frac{A}{(2x-3)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$

5. To every quadratic factor of the form ax^2+bx+c in the denominator, there will be P.F. of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where A and B are constants to be determined.

Example:
$\frac{x^2+x+1}{(x^2-1)(x+3)}$ will have its P.F’s of the form $=\frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{C}{(x+3)}$

	Factor in denominator	Term in partial fraction decomposition
1.	$ax+b$	$\frac{A}{ax+b}$
2.	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+........+\frac{A_k}{(ax+b)^k}$, k= 1, 2, 3,…..
3.	ax^2+bx+c	$\frac{Ax+B}{(ax^2+bx+c)}$
4.	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+........+\frac{A_Kx+B_K}{(ax^2+bx+c)^k}$

1.6 RESOLUTION INTO PARTIAL FRACTION

There are several types of resolution into partial fractions

Type 1:

Denominator containing linear factors.

When the denominator has all the factors linear

Example

$\frac{f(x)}{(x-a_1)(x-a_2)......(x-a_n)}$, where $f(x)$ is a polynomial of degree less than n.

Type 2:

Denominator containing a factor of the form $(x-a)^n$.

When the denominator has linear repeated factors.

Example

$\frac{f(x)}{(x-a)^n}$, where $f(x)$ is a polynomial of degree less than n.

Type 3:

Denominator containing a quadratic factor of the form $ax^2 + bx + c$ (where a, b, c are constants)

When the denominator has non – repeated quadratic factors.

$\frac{f(x)}{(ax^2+bx+c)(dx^2+ex+g)}$, where $f(x)$ is a polynomial of degree less than 2.

Type 4:

Denominator containing repeated quadratic factor of the form $ax^2 + bx + c$ (where a, b, c are constants)

When the denominator has repeated quadratic factors.

$\frac{f(x)}{x\,(ax^2+bx+c)(ax^2+bx+c)}$, where $f(x)$ is a polynomial of degree less than 2.

Type 1

The partial fraction is $\frac{f(x)}{(x-a_1)(x-a_2)......(x-a_n)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + ... + \frac{A_n}{(x-a_n)}$,

where $A_1, A_2, \ldots\ldots\ldots, A_n$ are constants.

Example 1

Resolve into partial fractions $\frac{3x+11}{(x-3)(x+2)}$

Solution

To get the form of the partial fraction,

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)} \quad \text{-------}(1)$$

add the right side, $\frac{3x+11}{(x-3)(x+2)} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$

choose *A* and *B*

$$3x + 11 = A(x + 2) + B(x - 3)$$

Choose $x = -2$ and $x = 3$.

we get,

Put $x = -2$, $5 = A(0) + B(-5) \Rightarrow \underline{B = -1}$

Put $x = 3$, $20 = A(5) + B(0) \Rightarrow \underline{A = 4}$

Substituting the value of A and B, The equation (1) becomes

$$\frac{3x+11}{(x-3)(x+2)} = \frac{4}{(x-3)} - \frac{1}{(x+2)}$$

Example 2

Resolve into partial fractions $\frac{2x+3}{(x-1)(x-2)(x-3)}$

Solution

Let $\frac{2x+3}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$ --------(1)

$2x+3 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

Put $x = 1$, $5 = 2A \Rightarrow$ $\underline{A = 5/2}$

Put $x = 2$, $7 = -B \Rightarrow$ $\underline{B = -7}$

Put $x = 3$, $9 = 2C \Rightarrow$ $\underline{C = 9/2}$

Substituting A, B and C in (1)

$$\frac{2x+3}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{7}{(x-2)} + \frac{9}{2(x-3)}$$

Example 3

Resolve into partial fractions $\frac{x^2+x+1}{(x^2-1)(x+3)}$

Solution

$$\frac{x^2+x+1}{(x^2-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}$$

$x^2 + x + 1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$

Put $x = 1$, $3 = A(2)(4) \Rightarrow$ $\underline{A = 3/8}$

Put $x = -1$, $1 = B(-2).2 \Rightarrow$ $\underline{B = -1/4}$

Put $x = -3$, $7 = C(-4).(-2) \Rightarrow$ $\underline{C = 7/8}$

Substituting A, B and C in (1)

$$\frac{x^2+x+1}{(x^2-1)(x+3)} = \frac{3}{8(x-1)} - \frac{1}{4(x+1)} + \frac{7}{8(x+3)}$$

PROBLEMS FOR PRACTICE

Resolve the following problems into partial fractions

1. $\dfrac{5x+2}{(1+3x)(1+2x)}$ answer : $\dfrac{1}{(1+3x)}+\dfrac{1}{(1+2x)}$

2. $\dfrac{x^2+x+1}{(x-1)(x-2)(x-3)}$ answer $\dfrac{3}{2(x-1)}-\dfrac{7}{(x-2)}+\dfrac{13}{2(x-3)}$

3. $\dfrac{3-7x^2}{(1-3x)(1+2x)(1+x)}$ answer : $\dfrac{1}{(1-3x)}+\dfrac{1}{(1+2x)}+\dfrac{1}{(1+x)}$

4. $\dfrac{x}{(x-1)(2x+1)}$ answer : $\dfrac{1}{3(x-1)}+\dfrac{1}{3(2x+1)}$

5. $\dfrac{6x^2-7x-3}{(x-1)(x-2)(x-3)}$ answer $\dfrac{2}{(x-1)}+\dfrac{3}{(x-3)}-\dfrac{4}{(2x-1)}$

Type II

The partial fractions are $\dfrac{A_1}{(x-a)}+\dfrac{A_2}{(x-a)^2}+........+\dfrac{A_n}{(x-a)^n}$

Example 4

Resolve into partial fractions $\dfrac{x^2+1}{(2x-3)(x-1)^2}$

Solution

$$Let \quad \frac{x^2+1}{(2x-3)(x-1)^2}=\frac{A}{(2x-3)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$$

x^2+1 = A(x-1)^2 + B(2x-3)(x-1) + C(2x-3)

Put x = 3/2, 13/4 = A(1/4) $\Rightarrow$ <u>A = 13</u>

Put x = 1, 2 = -C $\Rightarrow$ <u>C = -2</u>

Equating the coefficient of x^2, 1= A + 2B, 2B = 1- 13 = -12 $\Rightarrow$ <u>B = -6</u>

Substituting A, B and C in (1)

$$\frac{x^2+1}{(2x-3)(x-1)^2}=\frac{13}{(2x-3)}-\frac{6}{(x-1)}-\frac{2}{(x-1)^2}$$

Example 5

Resolve into partial fractions $\dfrac{3x^3 - 8x^2 + 10}{(x-1)^4}$

Solution

Here is the another method, it is easier than the regular method.

Put $x - 1 = y \Rightarrow x = y + 1$

$$\frac{3x^3 - 8x^2 + 10}{(x-1)^4} = \frac{3(y+1)^3 - 8(y+1)^2 + 10}{(y)^4}$$

$$= \frac{3y^3 + y^2 - 7y + 5}{(y)^4}$$

$$= \frac{3}{y} + \frac{1}{(y)^2} - \frac{7}{y^3} + \frac{5}{(y)^4}$$

$$= \frac{3}{(x-1)} + \frac{1}{(x-1)^2} - \frac{7}{(x-1)^3} + \frac{5}{(x-1)^4}$$

Type III

$$\frac{f(x)}{(ax^2 + bx + c)} = \frac{Ax + B.........}{(ax^2 + bx + c.......}$$

Example 6

Resolve into partial fractions $\dfrac{x^2}{(x^2+1)(x^2+2)(x^2+3)}$

Solution

Let $x^2 = y$

$$\frac{x^2}{(x^2+1)(x^2+2)(x^2+3)} = \frac{y}{(y+1)(y+2)(y+3)}$$

$$\frac{y}{(y+1)(y+2)(y+3)} = \frac{A}{(y+1)} + \frac{B}{(y+2)} + \frac{C}{(y+3)}$$

y = A(y+2)(y+3) + B(y+1)(y+3) + C(y+1)(y+2)

Put y = -1, -1 = A(-1)(2) $\Rightarrow$ A = -1/2

Put y = -2, -2 = B(-1).1 $\Rightarrow$ B = 2

Put y = -3, -3 = C(-2).(-1) $\Rightarrow$ C = -3/2

Substituting A, B and C in (1)

$$\frac{y}{(y+1)(y+2)(y+3)} = \frac{-1}{2(y+1)} + \frac{2}{(y+2)} - \frac{3}{2(y+3)}$$

$$\frac{x^2}{(x^2+1)(x^2+2)(x^2+3)} = \frac{-1}{2(x^2+1)} + \frac{2}{(x^2+2)} - \frac{3}{2(x^2+3)}$$

Example 7

Resolve into partial fractions $\frac{14x^3+14x^2-4x+3}{(3x^2-x+1)(x-1)(x+2)}$

Solution

$$\frac{14x^3+14x^2-4x+3}{(3x^2-x+1)(x-1)(x+2)} = \frac{Ax+B}{(3x^2-x+1)} + \frac{C}{(x-1)} + \frac{D}{(x+2)}$$

$14x^3+14x^2-4x+3 = (Ax+B)(x-1)(x+2)$ + B(x-1)(x+2) + C(x+2) $(3x^2-x+1)$ +D (x - 1) $(3x^2-x+1)$

Here we have two linear factors and hence two constants can be determined easily.

Put x = 1, 27 = 9C $\Rightarrow$ C = 3.

Put x = -2, -45 = -45D $\Rightarrow$ D = 1

Compare the coefficients of x^3 and constant term.

14 = A + 3C + 3D $\Rightarrow$ A = 2.

Comparing constant term,

3 = -2B + 2C –D $\Rightarrow$ B = 1.

$$\frac{14x^3+14x^2-4x+3}{(3x^2-x+1)(x-1)(x+2)} = \frac{2x+1}{(3x^2-x+1)} + \frac{3}{(x-1)} + \frac{1}{(x+2)}$$

Type IV

When the denominator has repeated quadratic factor

Example 8

Resolve into partial fractions $\dfrac{2x^4+2x^2+x+1}{x(x^2+1)^2}$

Solution

$$\frac{2x^4+2x^2+x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)}$$

$$2x^4+2x^2+x+1 \equiv A(x^2+1)^2 + (Bx+C)x + (Ex+F)x(x^2+1)$$

There is one linear factor x here. So put x = 0 in both sides of the identity.

1 = A

Compare the coefficients x^4 , x^3, x^2 and constant term.

Comparing,

x^4 , A + E = 2 => E = 1

x^3, F = 0

x^2, 2A + B + E = 0 => B = -1

x, we get C + F = 1 => C = 1

Partial fractions,

$$\frac{2x^4+2x^2+x+1}{x(x^2+1)^2} = \frac{1}{x} + \frac{1-x}{(x^2+1)^2} + \frac{x}{(x^2+1)}$$

PROBLEMS FOR PRACTICE

Resolve into partial fractions

1. $\dfrac{7x-1}{1-5x+6x^2}$ answer : $\dfrac{4}{(1-3x)} - \dfrac{5}{(1-2x)}$
2. $\dfrac{x^2-10x+13}{(x-1)(x^2-5x+6)}$ answer $\dfrac{2}{(x-1)} + \dfrac{3}{(x-2)} + \dfrac{4}{(x-3)}$
3. $\dfrac{9}{(x-1)(x+2)^2}$ answer : $\dfrac{1}{(x-1)} - \dfrac{1}{(x+2)} - \dfrac{3}{(x+2)^2}$
4. $\dfrac{2x+1}{(x-1)(x^2+1)}$ answer : $\dfrac{3}{2(x-1)} - \dfrac{3x-1}{2(x^2+1)}$

5. $\dfrac{3x^3 - 8x^2 + 10}{(x-1)^4}$ answer $\dfrac{3}{(x-1)} + \dfrac{1}{(x-1)^2} - \dfrac{7}{(x-1)^3} + \dfrac{5}{(x-1)^4}$

Determine the partial fraction decomposition of each of the following.

6. $\dfrac{8x-42}{(x^2+3x-18)}$ answer $\dfrac{10}{(x+6)} - \dfrac{2}{(x-3)}$

7. $\dfrac{9-9x}{(2x^2+7x-4)}$ answer $\dfrac{1}{(2x-1)} - \dfrac{5}{(x+4)}$

8. $\dfrac{4x^2}{(x-1)(x-2)^2}$ answer $\dfrac{4}{(x-1)} + \dfrac{16}{(x-2)^2}$

9. $\dfrac{9x+25}{(x+3)^2}$ answer $\dfrac{9}{(x+3)} - \dfrac{2}{(x+3)^2}$

Now, we need to do a set of exercise with quadratic factors

10. $\dfrac{8x^2-12}{x(x^2+2x-6)}$ answer $\dfrac{2}{x} + \dfrac{6x-4}{(x^2+2x-6)}$

11. $\dfrac{3x^3+7x-4}{(x^2+2)^2}$ answer $\dfrac{3x}{(x^2+2)} + \dfrac{x-4}{(x^2+2)^2}$

Chapter 2
Summation of Series

Contents

2.0 AIMS AND OBJECTIVES

Our Aim is to learn about the **Summation of Series**. Further we aim at learning summation of binomial, logarithmic and exponential series and the coefficients of its nth term.

2.1 SUMMATION OF SERIES

Vandermonde's theorem- Binomial theorems - Binomial theorem for rational index- Particular cases of the Binomial expansion- Sign of terms in the Binomial expansion- Application of the Binomial theorem to the summation of series- Sum of the coefficients – Approximate values - Exponential series - Logarithmic series - Convergence and divergence of series - Conditionally convergent series.

2.2. VANDERMONDE'S THEOREM

If r is a positive integer, then

$(m + n)_r = m_r + rC_1.m_{r-1}n_1 + rC_2.m_{r-2}n_2 + \ldots\ldots + n_r$ for all values of m and n.

If m and n are positive integers, we have

$$(1+x)^m = 1 + \frac{m_1}{1!}x + \frac{m_2}{2!}x^2 + \ldots\ldots + \frac{m_r}{r!}x^r + \ldots\ldots \text{-------}(1)$$

$$(1+x)^n = 1 + \frac{n_1}{1!}x + \frac{n_2}{2!}x^2 + \ldots\ldots + \frac{n_r}{r!}x^r + \ldots\ldots \text{--------}(2)$$

The product of two series (1) and (2) is

$$(1 + x)^{m+n} = 2 + (\frac{m_1}{1!} + \frac{n_1}{1!})x + (\frac{m_2}{2!} + \frac{n_2}{2!})x^2 + \ldots\ldots + \frac{m_{r-1}}{(r-1)!}.\frac{n_1}{1!} + \frac{m_{r-2}}{(r-2)!}.\frac{n_2}{2!} + \ldots$$

$$\ldots\ldots + \frac{n_{r-1}}{(r-1)!}.\frac{m_1}{1!} + \frac{n_r}{r!}$$

The coefficient of x^r in the product of the two series

$$= \frac{m_r}{r!} + \frac{n_r}{r!} + \frac{m_{r-1}}{(r-1)!}.\frac{n_1}{1!} + \frac{m_{r-2}}{(r-2)!}.\frac{n_2}{2!} + \ldots\ldots + \frac{n_{r-1}}{(r-1)!}.\frac{m_1}{1!} + \frac{n_r}{r!}$$

$$= \frac{1}{r!}\{m_r + \frac{r!}{(r-1)!.1!}.m_{r-1}.n_1 + \frac{r!}{(r-2)!.2!}.m_{r-2}.n_2 + \ldots\ldots + \frac{r!}{1!.(r-1)!}.m_1.n_{r-1} + n_r\}$$

$$= \frac{1}{r!}\{m_r + rc_1.m_{r-1}.n_1 + rc_2.m_{r-2}.n_2 + \ldots\ldots + rc_{r-1}.m_1.n_{r-1} + n_r\}$$

= Coefficient of x^r in $(1+x)^{m+n}$

$$= \frac{(m+n)_r}{r!}$$

Therefore $(m + n)_r = m_r + rC_1.m_{r-1}n_1 + rC_2.m_{r-2}n_2 + \ldots\ldots + n_r$

Each side is a polynomial in m and n of the r^{th} degree. This result is true for all integral values of m and n.

i.e., the result is true for more than r values of m and n.
Hence the polynomials are identically equal,
i.e., the result is true for all values of m and n.

2.3 BINOMIAL THEOREMS

2.3.1 BINOMIAL THEOREM FOR RATIONAL INDEX

If n is a rational number and -1 < x < 1, i.e., $|x| < 1$ the sum of the series

$$1+nx + \frac{n(n-1)}{2!}x^2 + ...+ \frac{n(n-1)......(n-r+1)}{r!}x^r + ...$$

is the real positive value of $(1 + x)^{n.}$
Represent the sum of the series by $f(n)$.

Then $f(n) = 1 + \frac{n_1}{1!}x + \frac{n_2}{2!}x^2 + ...+ \frac{n_r}{r!}x^r + ...$

We can sum this series only when it is absolutely convergent.
Denoting this series by $u_1 + u_2 + u_3 + ...$

$$\frac{u_{r+1}}{u_r} = \frac{n_r}{r!}x^r + \frac{n_{r-1}}{(r-1)!}x^{r-1} = \frac{n-r+1}{r}x$$

Therefore $\left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{n-r+1}{r}\right||x| = \left|\frac{n+1}{r} - 1\right||x|$

Therefore $\lim_{x\to\infty} \frac{u_{r+1}}{u_r} = |x|$

Therefore the series $|u_1| + |u_2| + ... + |u_r| + ...$ Is convergent if $|x| < 1$.
i.e., the series $u_1 + u_2 + u_3 + ...$ is absolutely convergent if $|x| < 1$.
Similarly the series

$$f(m+n) = 1 + \frac{(m+n)_1}{1!}x + \frac{(m+n)_2}{2!}x^2 + ...+ \frac{(m+n)_r}{r!}x^r + ...$$

are also absolutely convergent if $|x| < 1$.
$f(m)$ and $f(n)$ can be multiplied and the resulting series is also an absolutely convergent series.
The coefficient of x^r in the product of $f(m)$ and $f(n)$ is

$$\frac{m_r}{r!}x + \frac{m_{r-1}}{(r-1)!}.\frac{n_1}{1!} + \frac{m_{r-2}}{(r-2)!}.\frac{n_2}{2!} + ...+ \frac{n_r}{r!}$$

$$\frac{1}{r!}\{m_r + rC_1.m_{r-1}n_1 + rC_2.m_{r-2}n_2 + ...+ u_r\}.$$

By Vandermande's Theorem the expression is equal to $\frac{(m+n)_r}{r!}$ which is the coefficient of x^r in $f\left(m+n\right)$.

Thus the coefficient of any power of x in $f\left(m+n\right)$ is equal to the coefficient of the same power of x in the product $f(m).f(n)$.

$f(m).f(n) = f\left(m+n\right)$ for all values of m and n, provided that x is numerically less than unity.

Hence $f(m).f(n).f(p) = f\left(m+n\right).f(p) = f\left(m+n+p\right)$.

Therefore $f(m).f(n).f(p).........s$ factors $= f\left(m+n+p+.......+s \text{ terms}\right)$

Case i. n is a positive fraction.

Let n be equal to $\frac{r}{s}$ where r and s are positive integers.

$f(r) = (1+x)^r$ since r is a positive integer.

Therefore $f(0) = 1$ and $f(1) = 1+x$.

$f(\frac{r}{s}).f\left(\frac{r}{s}\right)$.......$s$ factors = $f(\frac{r}{s}+\frac{r}{s}+........+s$ terms)

i.e., $\{ f\left(\frac{r}{s}\right)\}^s = f(r)$

$= (1+x)^r$ since r is a positive integer.

$f(\frac{r}{s}) = (1+x)^{\frac{r}{s}}$

Therefore $f(n) = (1+x)^n$

This proves the Binomial Theorem for a positive fraction index. Therefore the theorem is true for any positive index.

Case ii. n is a negative rational number.

Let n = $-m$ where m is a positive rational number.

$f(m).f(n) = f(m+n) = f(0) = 1$

$f(n) = \frac{1}{f(m)} = \frac{1}{(1+x)^m}$ since m is positive

$= (1+x)^{-m} = (1+x)^n$

Hence the theorem is true for any negative index.

2.3.2 BINOMIAL SERIES

When $-1 \prec x \prec 1 (or) |x| \prec 1$

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ... + \frac{n(n-1)(n-2).....(n-(r-1))}{1.2.3.....r}x^r + ...\infty$
2. $(1-x)^n = 1 - nx + \frac{n(n-1)}{1.2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}x^3 + ... + (-1)^r\frac{n(n-1)(n-2).....(n-(r-1))}{1.2.3.....r}x^r + ...\infty$
3. $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{1.2}x^2 - \frac{n(n+1)(n+2)}{1.2.3}x^3 + ... + (-1)^r\frac{n(n+1)(n+2).....(n+(r-1))}{1.2.3.....r}x^r + ...\infty$
4. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + ... + \frac{n(n+1)(n+2).....(n+(r-1))}{1.2.3.....r}x^r + ...\infty$
5. $(1+x)^{\frac{p}{q}} = 1 + \frac{p}{1!}(\frac{x}{q}) + \frac{p(p-q)}{2!}(\frac{x}{q})^2 + \frac{p(p-q)(p-2q)}{3!}(\frac{x}{q})^3 + ... + \frac{p(p-q)(p-2q)....(p-(r-1)q)}{r!}(\frac{x}{q})^r + ...\infty$
6. $(1-x)^{\frac{p}{q}} = 1 - \frac{p}{1!}(\frac{x}{q}) + \frac{p(p-q)}{2!}(\frac{x}{q})^2 - \frac{p(p-q)(p-2q)}{3!}(\frac{x}{q})^3 + ... + (-1)^r\frac{p(p-q)(p-2q)....(p-(r-1)q)}{r!}(\frac{x}{q})^r + ...\infty$

7. $(1+x)^{-\frac{p}{q}} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \dots + (-1)^r \frac{p(p+q)(p+2q)\dots(p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \dots$

8. $(1-x)^{-\frac{p}{q}} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \dots + \frac{p(p+q)(p+2q)\dots(p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \dots$

9. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-1)^r x^r + \dots$

10. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots + (-1)^r x^r + \dots\infty$

11. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots + (-1)^r (r+1)x^r + \dots$

12. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots + (r+1)x^r + \dots\infty$

13. $(1-x)^{-3} = \frac{1}{2}\{1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \dots + (n+1)(n+2)x^n + \dots\}$

14. $(1-x)^{-4} = \frac{1}{6}\{1.2.3 + 2.3.4x + 3.4.5x^2 + 4.5.6x^3 + \dots + (n+1)(n+2)(n+3)x^n + \dots\}$

15. $(1-x)^{\frac{-1}{2}} = 1 + \frac{1}{2}x + \frac{1}{2}.\frac{3}{4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \dots$

16. $(1-x)^{\frac{-1}{3}} = 1 + \frac{1}{3}x + \frac{1}{3}.\frac{4}{6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \dots$

2.3.3BINOMIAL THEOREM

Example: $a+b$ is a binomial (the two terms are **a** and **b**)
Let us multiply $(a+b)$ by itself using Polynomial Multiplication :

$(a+b)(a+b) = a^2 + 2ab + b^2$

Now take that result and multiply by $(a+b)$ again:

$(a^2 + 2ab + b^2)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$

And again: $(a^3 + 3a^2b + 3ab^2 + b^3)(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

The calculations get longer and longer as we go, but there is some kind of **pattern** developing.

That pattern is summed up by the **Binomial Theorem**: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Let's look at **all the results** we got before, from $(a+b)^0$ up to $(a+b)^3$ from

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

1

$1a \quad 1b$

$1a^2 \quad 2ab \quad 1b^2$

$1a^3 \quad 3a^2b \quad 3ab^2 \quad 1b^3$

Combinations and Permutations, it matches to pascal's Triangle like:

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Let's see if the formula works:

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4.3.2.1}{2.1.2.1} = 6$$

So let's try using it for $n = 3$: in binomial

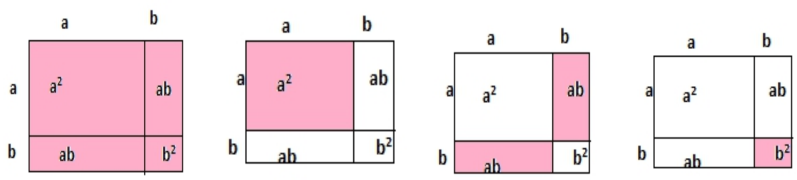
$$(a+b)^3 = \sum_{k=0}^{3} \binom{3}{k} a^{3-k} b^k$$

$$= \binom{3}{0} a^{3-0} b^0 + \binom{3}{1} a^{3-1} b^1 + \binom{3}{2} a^{3-2} b^2 + \binom{3}{3} a^{3-3} b^3$$

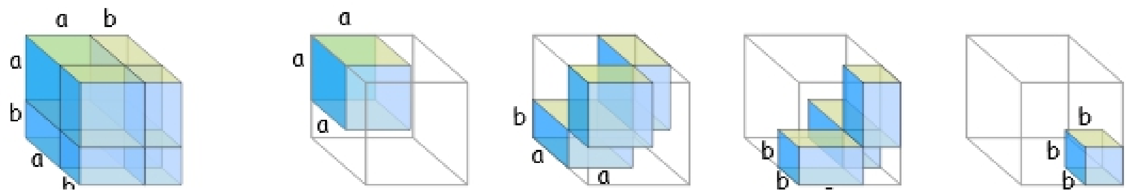
$$= 1.a^3.1 \ + \ 3.a^2.b \ + \ 3.a.b^2 \ + \ 1.b^3$$

$$= \ a^3 \ + \ 3.a^2.b \ + \ 3.a.b^2 \ + \ b^3$$

The Binomial Theorem can be shown using Geometry:
In 2 dimensions, $(a+b)^2 = a^2 + 2ab + b^2$



In 3 dimensions,
$(a+b)^3 \ = a^3 + 3a^2b + 3ab^2 + b^3$



2.3.4 PROBLEMS USING THE BINOMIAL THEOREM

1. *Use the binomial theorem to find the* 7^{th} *power of* 11

Solution

$11^7 = (10 + 1)^7$

$= 10^7 + 7C_1.10^6 + 7C_2.10^5 + 7C_3.10^4 + 7C_4.10^3 + 7C_5.10^2 + 7C_6.10 + 7C_7$

$= 10^7 + 7.10^6 + 21.10^5 + 35.10^4 + 35.10^3 + 21.10^2 + 7.10 + 1$

$= 10000000 + 7000000 + 2100000 + 350000 + 35000 + 2100 + 70 + 1$

$= 19,487,171$

2. *Show that* $2^{2n} - 3n - 1$ *is divisible by* 9 *for all positive integral values of n*

Solution

$2^{2n} - 3n - 1 = (2^2)^n - 3n - 1$

$= 4^n - 3n - 1$

$= (3+1)^n - 3n - 1$

$= 3^n + nC_1.3^{n-1} + nC_2.3^{n-2} + \ldots\ldots + nC_{n-1}.3 + 1 - 3n - 1$

$= 3^n + nC_1.3^{n-1} + nC_2.3^{n-2} + \ldots\ldots + 3n + 1 - 3n - 1$

$= 3^n + nC_1.3^{n-1} + nC_2.3^{n-2} + \ldots\ldots + nC_{n-2}.3^2$

32 , i.e., 9 is a factor in each term.
22 n - 3 n - 1 is exactly divisible by 9.

3. Prove that if is a positive integer the integral part of $(4 + \sqrt{10})^n$ is an odd number.

Solution

Let $(4 + \sqrt{10})^n$ be I + F, where I is an integer and F a proper fraction.

Consider the expression $(4 + \sqrt{10})^n$

$\sqrt{10}$ lies between 3 and 4.

Therefore $(4 + \sqrt{10})$ is a fraction when n is a positive integer and let it be α.

$(4 + \sqrt{10})^n + (4 - \sqrt{10})^n$

$= 4^n + nC_1.4^{n-1}.\sqrt{10} + nC_2.4^{n-2}.(\sqrt{10})^2 + \ldots + (\sqrt{10})^n + 4^n - nC_1.4^{n-1}\sqrt{10} + nC_2.4^{n-2}.(\sqrt{10})^2 - \ldots$

$= 2\{4^n + nC_2.4^{n-2}.(\sqrt{10})^2 + nC_4.4^{n-4}(\sqrt{10})^4 + \ldots\ldots\}$

The expansion on the right contains only even powers of $\sqrt{10}$ and since the binomial coefficient are all integers, we get that expression within the brackets is an integer.

Hence we get the result

$I + F + \alpha = 2(\text{an integer})$

$I + F + \alpha = \text{an even number}$

Therefore $F + \alpha = \text{an even integer} - 1$

$= \text{an integer.}$

F and α are proper fractions

Therefore $F + \alpha < 2$

$\text{Hence}\, F + \alpha = 1$

$I + 1 = \text{an even integer}$

$I = \text{an odd integer}$

$4.$ *If I is the integral part and F the fractional part of* $(3\sqrt{3}+5)^{2n+1}$, *where n is a positive integer, show that* $F(I + F) = 2^{2n+1}$

Solution

$(3\sqrt{3} + 5)^{2n+1} = I + F$ (Given)

Let us consider the expression $(3\sqrt{3} + 5)^{2n+1}$ and let it be G.

$I + F = (3\sqrt{3}+5)^{2n+1} - (3\sqrt{3} - 5)^{2n+1}$

Write $a = 3\sqrt{3}$ and $b = 5$

$I + F = (a + b)^{2n+1} - (a - b)^{2n+1}$

$= a^{2n+1} + (2n + 1)C_1.a^{2n}.b + (2n + 1)C_2.a^{2n-1}.b^2 + \ldots\ldots. + b^{2n+1}$

$-\{ a^{2n+1} - (2n + 1)C_1.a^{2n}.b + (2n + 1)C_2.a^{2n-1}.b^2 + \ldots\ldots. - b^{2n+1} \}$

$= 2\{(2n + 1)C_1.a^{2n}.b + (2n + 1)C_2.a^{2n-1}.b^3 + \ldots\ldots.\}$

Since binomial coefficients in the expansion are integers and since only even powers of a, i.e., $3\sqrt{3}$ occur in the expansion we get that

$I + F - G = 2(\text{an integer})$

$= \text{an even number}$

$F - G = \text{an even number} - \text{an integer} = \text{intege r}$

F is a proper fraction $\sqrt{3} < 2$

$3\sqrt{3} < 6$ and also $\sqrt{3} = 1.7\ldots.$

$0 < (3\sqrt{3} - 5) < 1$

G i.e., $(3\sqrt{3} - 5)2n + 1$ is a proper fraction

$F - G < 1$

But we have proved that $F - G =$ an integer

$F = G$

Then $F(I + F) = G(I + F)$

$= (3\sqrt{3} - 5)^{2n+1} (3\sqrt{3} - 5)^{2n+1} = (27 - 25)^{2n+1} = 2^{2n+1}$

2.3.5 PROBLEMS FOR PRACTICE

1. Write down the expansion of :

i. $(3x + 5y)^5$

Answer $243x^5 + 2025x^4y + 6750x^3y^2 + 11250x^2y^3 + 9375xy^4 + 3125y^5$

ii. $(3x^2 + \frac{4}{x})^6$

Answer $729x^{12} + 5832x^9 + 19440x^6 + \frac{4096}{x^6} + 34560x^3 + \frac{18432}{x^3} + 34560$

iii. $\left(5x-\frac{7}{x}\right)^5$ Answer $3125x^5-\frac{16807}{x^5}-21875x^3+\frac{60025}{x^3}+61250x-\frac{85750}{x}$

iv. $(8x^2-\frac{7}{x})^3$ Answer $512x^6-1344x^3-\frac{343}{x^3}+1176$

2. If $(6+\sqrt{35})^n = I + F$, where n and I are positive integers and F a positive fraction, prove that I is an odd integer and $(I-F)(I+F) = 1$.
3. If I is the integral part and F the fractional part of $(5\sqrt{2}+7)^{2n+1}$, show that $F(I+F) = 1$, n being a positive integer.

4. Show that $2^{3n+4} - 34n - 16$ is divisible by 289 for all positive integral values of n.

5. Show that $3^{2n} - 26n - 1$ is divisible by 676 for all positive integral values of n.

6. Show that $49^n + 16n - 1$ is divisible by 64 for all positive integral values of n.

2.3.6 SIGN OF TERMS IN THE BINOMIAL EXPANSION

Denoting the r^{th} term by u_r, we get

$$\frac{U_{r+1}}{U_r} = \frac{n-r+1}{r}x = -(1-\frac{n+1}{r})x$$

If $x > 0$ and $r > n+1$, $\frac{U_{r+1}}{U_r}$ is positive

If $x < 0$ and $r > n+1$, $\frac{U_{r+1}}{U_r}$ is positive

After a certain stage, the terms are alternately positive and negative if x is positive and are all of the same sign if x is negative.

2.3.7 FIND THE GREATEST TERM IN THE BINOMIAL EXPANSION

To find the greatest term in $(x+a)^n$, write $(x+a)^n = a^n(1+\frac{x}{a})^n$

$$= a^n(1+y)^n, \text{ where } y = \frac{x}{a}$$

The factor a^n, being a constant, will not affect the relative magnitude of the terms.

2.3.8 PROBLEMS ON GREATEST TERM IN THE BINOMIAL EXPANSION

1. *Find the greatest term in the expansion of* $(3x+2)^{35}$ *when* $x = 2$.

Solution

$(3x + 2)^{35} = 2^{35}(1+ \frac{3x}{2})^{35}$

It will be sufficient to consider the expansion of $(1+ \frac{3x}{2})^{35}$

Here $\frac{U_{r+1}}{U_r} = \frac{n-r+1}{r}.\frac{3x}{2}$ where n = 35, x = 2

$= \frac{35-r+1}{r}.3 \quad = \frac{108-3r}{r}$

$\frac{U_{r+1}}{U_r} \geq 1$ according as $\frac{108-3r}{r} \geq 1$

i.e., as $108 - 3r \geq r$
i.e., as $4r \leq 108$
i.e., as $r \leq 27$.
Hence for all values of r up to 26, we have $U_{r+1} \geq U_r$ but for r = 27, we get $U_{r+1} = U_r$ and for values of $r > 27$, we get $U_{r+1} < U_r$

Thus the 27th and 28th terms are numerically equal and are greater than any other term. Their values are each equal to $2^{35}. 3^8. 35C_{27}$.

2. *Find the greatest term in the expansion of* $(2x - 3)^9$ *when* $x = 3$.

Solution

$(2x - 3)^9 = (-3)^9 (1 - \frac{2x}{3})^9$

Here $\frac{U_{r+1}}{U_r} = \frac{9-r+1}{r}.\frac{2x}{3}$

$= \frac{9-r+1}{r}.2 \quad = \frac{20-2r}{r}$

$|\frac{U_{r+1}}{U_r}| \geq 1$ according as $\frac{20-2r}{r} \geq 1$

i.e., as $20 - 2r \geq r$
i.e., as $3r \leq 20$
i.e., as $r \leq 6\frac{2}{3}$.

Thus the greatest term is the 7 th and its value = $(-3)^9. 9C_6. . (-\frac{2x}{3})^6 = - 9C_6 (3)^9. 2^6$.

2.3.9 PROBLEMS FOR PRACTICE

1. Find the greatest expansion of the following:
   i . $(1-\frac{2}{3})^{14}$ Answer $2^6/ 3^6. 14C_6$. t_7 is the greatest term
   ii. $(5x + 4)^7$ when x = 1 Answer $5^3/ 4^4. 7C_6$. T_5 is the greatest term
   iii. $(2 + 3x)^8$ when $x = \frac{2}{3}$ Answer $2^8. 3C_4$. T_5 is the greatest term

2.3.10 FIND THE MIDDLE TERM OF THE BINOMIAL EXPANSION

1. *Write down the middle term of* $\left(\frac{y\sqrt{x}}{5}-\frac{5}{x\sqrt{y}}\right)^{12}$

Solution

Since there are 13 terms in the expansion, the middle term is the 7^{th} .

General term $U_{r+1} = 12C_r.\left(\frac{y\sqrt{x}}{5}\right)^{12-r}\left(-\frac{5}{x\sqrt{y}}\right)^{r}$

Putting r = 6.

$U_7 = 12C_6.\left(\frac{y\sqrt{x}}{5}\right)^{6}\left(-\frac{5}{x\sqrt{y}}\right)^{6}$

$= 12C_6.\frac{y^6x^3}{5^6}.\frac{5^6}{x^6y^3}$

$= 12C_6.\frac{y^3}{x^3}$

2. *Write down the middle terms in the expansion of* $\left(2x-\frac{3}{x}\right)^{15}$

Solution

Since there are the 16 terms in the expansion, the 8^{th} and the 9^{th} terms are the middle terms.

General term $U_{r+1} = 15\,C_r\,.\,(2x)^{15-r}.\left(\frac{-3}{x}\right)^{r}$

Putting r = 7, $U_8 = 15C_7.\,(2x)^8.\left(\frac{-3}{x}\right)^{7}$

$= -\,15C_7.\,2^8.\,3^7.\,X.$

Putting r = 8, $U_9 = 15C_8.\,(2x)^7.\left(\frac{-3}{x}\right)^{8}$

$= 15C_8.\,2^7.\,3^8.\,\frac{1}{x}.$

3. *Find the term independent of x in the expansion of* $(3x^{\frac{1}{2}}-3x^{\frac{-1}{3}})^{20}$

Solution

Let $(r+1)^{th}$ term contain no x.

$U_{r+1} = 20C_r.\,(2x^{\frac{1}{2}})^{20-r}.(-3x^{\frac{-1}{3}})^{r}$

$= (-1)^r.\,(2)^{20-r}.3^r.20C_r.x^{10-\frac{5r}{6}}$

Since x does not occur in this term.

$10-\frac{5r}{6} = 0$ Therefore r = 12

Hence the 13^{th} term is independent of x

$U_{13} = (-1)^{13} . 2^3. 3^{12}. 20C_{12} = 2^8 . 3^{12}. 20C_{12}.$

4. *If a_r be the coefficient of x^r in the expansion of $(1+x)^n$, prove that*

$$\frac{a_1}{a_0}+2.\frac{a_2}{a_1}+3.\frac{a_3}{a_2}+ ....+n.\frac{a_n}{a_{n-1}} = \frac{n(n+1)}{2}$$

Solution

$U_{r+1} = nC_r. 1^{n-r}. x^r$

$= nC_r. x^r$

Hence $a_r = nC_r$

$$\frac{a_r}{a_{r-1}} = \frac{nC_r}{nC_{r-1}} = \frac{n!}{r!.(n-r)!}.\frac{(r-1)!.(n-r+1)!}{n!}$$

$$= \frac{n-r+1}{r}$$

$$r.\frac{a_r}{a_{r-1}} = n-r+1$$

Giving the values, 1, 2, 3, …..,n for r, we get

$1.\frac{a_1}{a_0} = n$

$2.\frac{a_2}{a_1} = n - 1$

$3.\frac{a_3}{a_2} = n - 2$ .... $n.\frac{a_n}{a_{n-1}} = 1$

$$\frac{a_1}{a_0}+2.\frac{a_2}{a_1}+3.\frac{a_3}{a_2}+ ....+n.\frac{a_n}{a_{n-1}} = n + (n-1) + (n-2)+.......+2+1 = \frac{n(n+1)}{2}$$

2.3.11 FIND THE GENERAL TERM OF THE BINOMIAL EXPANSION

1. *Find the general term of the expansion of* $(1+x)^{\frac{2}{3}}$

Solution

The $(r+1)^{th}$ term $= \dfrac{\frac{2}{3}\left(\frac{2}{3}-1\right)(\frac{2}{3}-2).......(\frac{2}{3}-r+1)}{r!}.x^r$

$$= \frac{2(-1)(-4)(-7)......(-3r+5)}{3^r.(r!)}.x^r$$

The number of factors in the numerator is r and (r- 1) of these are negative

The $(r+1)^{th}$ term $= (-1)^{r-1}\dfrac{2.1.4.7.....(3r-5)}{3^r.(r!)}x^r$

$2.\ \textit{Find the validity of the binomial expansion of} \left(2-3x\right)^{-1}$

Solution

$$(2 - 3x)^{-1} = \frac{1}{(2-3x)} = \frac{1}{2(1-\frac{3}{2}x)} = \frac{1}{2}(1 - \frac{3}{2}x)^{-1}$$

The expansion is valid if $|\frac{3}{2}x| < 1 \Rightarrow |x| < \frac{2}{3}$

2.3.12 FIND THE COEFFICIENT OF THE PARTICULAR TERMS IN THE BINOMIAL EXPANSION

$1.\ \textit{Find the coefficient of } x^{10} \textit{ in the expansion } \frac{1}{(2-3x)^2}$

Solution

$$\frac{1}{(2-3x)^2} = \frac{1}{4(1-\frac{3}{2}x)^2} = \frac{1}{4}(1-\frac{3}{2}x)^{-2}$$

$$\frac{1}{4}(1+2(\frac{3}{2})x + 3(\frac{3}{2}x)^2 + ...+ 11(\frac{3}{2}x)^{10} + ...\infty$$

Co efficient of $x^n = \frac{11}{4}(\frac{3^{10}}{2^{10}}) = 11.(\frac{3^{10}}{2^{12}})$

$2.\ \textit{If the first two terms in the expansion of } [\frac{a}{(1+2x)^2} + \frac{b}{(1-3x)}] \textit{ in ascending powers of } x \textit{ are } 3 \textit{ and } -5x \textit{ respectively, then find the coefficient of } x^n$

Solution

$S = a(1+2x)^{-2} + b(1-3x)^{-1}$

$= a[1-2(2x) + 3(2x)^2 + ...] + b[1 + 3x + (3x)^2 + ...]$

$= (a + b) + (-4a + 3b)x + ...+ x^n [(-1)^n (n + 1).a . 2^n + b. 3^n] + ...\infty$

$a + b = 3$ (given)------(1)

$-4a + 3b = -5$ ----------(2)

Solving (1) and (2), we get $a = 2$ and $b = 1$

Coefficient of $x^n = [(-1)^n (n + 1) . 2^n.2 + 1. 3^n] = \{(-1)^n (n + 1) . 2^{n+1} + 1. 3^n\}$

3. *Find the coefficient of* x^9 *in the expansion of* $\dfrac{1}{(1+x+x^2)}$

Solution

$(1 - x)(1 + x + x^2) = 1 + x + x^2 - x - x^2 - x^3 = 1 - x^3$

$$(1 + x + x^2) = \frac{1-x^3}{1-x}$$

$$(1 + x + x^2)^{-1} = \frac{1-x}{1-x^3}$$

$$\left(1 + x + x^2\right) = (1+x)(1 - x^3)^{-1}$$

$$= (1-x)\,[\,1 + x^3 + (x^3)^2 + (x^3)^3 + (x^3)^4 + \ldots\ldots]$$

$$= (1 + x)\,[1+x^3 + x^6 + x^9 + \ldots\ldots\ldots. + (x^3)^{n-1} + (x^3)^n + \ldots\ldots \infty\,]$$

$$= [1+x^3 + x^6 + x^9 + \ldots\ldots\ldots. + \ldots.\infty] - [x + x^4 + x^7 + \ldots..]$$

Coefficient of $x^9 = 1$

4. *Find the coefficient of* x^n *in the expansion of* $\left(1 - x + x^2 - x^3\right)^{-1}$

Solution

$(1 + x)(1 - x + x^2 - x^3) = 1 - x + x^2 - x^3 + x - x^2 + x^3 - x^4$

$$(1 - x + x^2 - x^3) = \frac{1-x^4}{1+x}$$

$$(1 - x + x^2 - x^3)^{-1} = (1+x)(1 - x^4)^{-1}$$

$$= (1 + x)\,[1+x^4 + (x^4)^2 + \ldots\ldots\ldots. + (x^4)^{n-1} + (x^4)^n + \ldots\ldots\infty\,]$$

$$= 1+x^4 + x^8 + \ldots\ldots\ldots. + x^{4n-4} + x^{4n} + \ldots\ldots. + x + x^5 + x^9 + \ldots.. + x^{4n-3} + x^{4n+1} + \ldots.\infty$$

Coefficient of $x^n = 1 + 0 = 1$

5. *Find the coefficient of* x^7 *in the expansion of* $\left(1 - x - x^2 + x^3\right)^6$

Solution

$(1 - x - x^2 + x^3) = (1 - x)(1 - x^2)$

$$(1 - x - x^2 + x^3)^6 = (1 - x)^6 (1 - x^2)^6$$

$$= 1 - 6C_1x + 6C_2x^2 - \ldots\ldots\ldots. + x^6).(1 - 6C_1\,.\,x^2 + 6C_2.x^4 - \ldots\ldots. + x^{12})$$

The required Coefficient of $x^7 = 6C_1\,.\,6C_3 - 6C_3\,.\,6C_2 + 6C_5\,6C_1$

$$= 6 \times 20 - 20 \times 15 + 6 \times 6 = -144.$$

6. *Find the coefficient of* x^{32} *in the expansion of* $(x^4 - \dfrac{1}{x^3})^{15}$

Solution

Let x^{32} occur in the $(r + 1)^{th}$ term in the expansion

$U_{r+1} = 15C_r. (x^4)^{15-r}. ((\frac{-1}{x^3})^r$

$U_{r+1} = 15C_r. (-1)^r. x^{60-7r}$

Therefore 60-7r = 32

r = 4

Hence the 5^{th} term contains x^{32}.

Co efficient of x^{32} in the expansion = $15C_4. (-1)^4 = 15C_4$

7. *Find the coefficient of the expansion of* x^n *in the expansion of* $\frac{1+3x+2x^2}{(1-x)^4}$ *in ascending powers of x.*

Solution

$$\frac{1+3x+2x^2}{(1-x)^4} = (1 + 3x + 2x^2)(1 - x)^{-4}$$

$$= (1 + 3x + 2x^2)\{ 1+4x+\frac{4.5}{2!}x^2+\frac{4.5.6}{3!}x^3+.......+\frac{(n+1)(n+2)(n+3)}{n!}x^n+.... \}$$

Coefficient of x^n $= \frac{(n+1)(n+2)(n+3)}{6}+3.\frac{n(n+1)(n+2)}{6}+\frac{2(n-1)n(n+1)}{6}$

$$=\frac{n+1}{6}\{6n^2+9n+6\} \quad =\frac{(n+1)(2n^2+3n+2)}{2}$$

8. *If n is a positive integer, prove that the coefficient of* x^{3n} *in the expansion of* $\frac{1+x}{(1+x+x^2)^3}$ *in a series of ascending powers of x is* $\frac{1}{2}(n+1)(3n+2)$

Solution

$$\frac{1+x}{(1+x+x^2)^3}=\frac{(1+x)(1-x)^3}{(1-x^3)^3}$$

$$=\left(1 - 2x + 2x^3 - x^4\right)\left(1 - x^3\right)^{-3}$$

$$=\left(1 - 2x + 2x^3 - x^4\right)\{1+3x^3+\frac{3.4}{2}x^6+......+\frac{(n+1)(n+2)}{n!}x^{3n}+....\}$$

Coefficient of x^{3n} $=\frac{(n+1)(n+2)}{2}+2.\frac{n(n+1)}{2} \quad =\frac{1}{2}(n+1)(3n+2)$

2.3.13 PROBLEMS FOR PRACTICE

1. *Find the coefficient of* x^8 *and* x^{10} *in the expansion of* $\frac{1}{(1+x+x^2)}$

Answer : Coefficient of $x^8 = 0$

Coefficient of $x^{10} = -1$

$2. \textit{If } n \textit{ is positive integer, prove that the coefficient of } x^n \textit{ in the expansion of } \frac{(3x-2)^n}{(1-x)^2}$

is $1 - 2n.$

3. Prove that if n is a positive integer, the coefficient of x^n in $\left(1 + x + x^2\right)^n$ is

$$1+.\frac{n(n+1)}{(1!)^2}+\frac{n(n-1)(n-2)(n-3)}{(2!)^2}+.....$$

4. Find the coefficient of x^n in the expansion of $\left(1+ 2x+ 3x^2+ \ldots\right)^2$

Answer $\frac{(n+1)(n+2)(n+3)}{6}$

5. Find the coefficient of x^n in the expansion of $\left(1 - 2x + 3x^2 - 4x^3 + \ldots.\right)^{-n}$

Answer $\frac{(2n)!}{4n!}$

6. Show that the coefficient of x^n in the expansion of $(1+ 2x)^3 (1 - x)^{-2}$ is $27(n - 1)$ for $n > 2$.

2.3.14 APPLICATION OF THE BINOMIAL SERIES TO THE SUMMATION OF SERIES

2.3.14.1 FIND THE SUM TO INFINITY

1. Find the sum of $1+2(\frac{1}{2})+3(\frac{1}{2})^2+4(\frac{1}{2})^3+.....\infty$

Solution

$S = 1+2(\frac{1}{2})+3(\frac{1}{2})^2+4(\frac{1}{2})^3+.....\infty$

Let $x = (\frac{1}{2})$

$=1+2x+3x^2+4x^3+.....\infty$

$= (1 - x)^{-2} = (1 - \frac{1}{2})^{-2} = (\frac{1}{2})^{-2} = 2^2 = 4$

2. Find the sum of the series $1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+.....\infty$

Solution

S $= 1+\frac{2}{1.6}+\frac{2.(2+3)}{1.6.12}+\frac{2.(2+3).(2+6)}{1.6.2.6.3.6}+.....\infty$

$=1+\frac{2}{1!}(\frac{1}{6})+\frac{2.(2+3)}{2!}(\frac{1}{6})^2+\frac{2.(2+3).(2+6)}{3!}(\frac{1}{6})^3+.....\infty$

where p = 2, q = 3:

$= (1-x)^{\frac{-p}{q}}$ = $(1 - \frac{1}{2})^{\frac{-2}{3}}$ = $(\frac{1}{2})^{\frac{-2}{3}}$ $\qquad \frac{x}{q} = \frac{1}{6} \Rightarrow x = \frac{3}{6} = \frac{1}{2}$

= $2^{\frac{2}{3}}$ = $(2^2)^{\frac{1}{3}}$ = $4^{\frac{1}{3}}$ = $\sqrt[3]{4}$

3. *Find the sum of* $1 + \frac{10}{9} + \frac{10.16}{9.18} + \frac{10.16.22}{9.18.27} + ...$

Solution

S $= (1-x)^{\frac{-p}{q}}$ where p = 10, q = 6: $\frac{x}{q} = \frac{1}{9} \Rightarrow x = \frac{6}{9} = \frac{2}{3}$

= $(1 - \frac{2}{3})^{\frac{-10}{6}}$ = $(\frac{1}{3})^{\frac{-5}{3}} = 3^{\frac{5}{3}}$

4. *Find the sum of* $1 - \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + ...$

Solution

S = $1 - \frac{1}{1.4} + \frac{1.(1+2)}{1.4.2.4} - \frac{1.(1+2).(1+4)}{1.4.2.4.3.4} + ...$

$=1 - \frac{1}{1!}(\frac{1}{4}) + \frac{1.(1+2)}{2!}(\frac{1}{4})^2 - \frac{1.(1+2).(1+4)}{1.4.2.4.3.4}(\frac{1}{4})^3 + ...$

S $= (1-x)^{\frac{-p}{q}}$ where p = 1, q = 2: $\frac{x}{q} = \frac{1}{4} \Rightarrow x = \frac{2}{4} = \frac{1}{2}$

S = $(1 + \frac{1}{2})^{\frac{-1}{2}}$ = $(\frac{3}{2})^{\frac{-1}{2}} = (\frac{2}{3})^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$

5. *Find the sum of* $1 - \frac{1}{5} + \frac{1.4}{5.10} - \frac{1.4.7}{5.10.15} + ...$

Solution

Similar to previous problem

S $= (1-x)^{\frac{-p}{q}}$ where p = 1, q = 3: $\frac{x}{q} = \frac{1}{5} \Rightarrow x = \frac{3}{5}$

S = $(1 + \frac{3}{5})^{\frac{-1}{3}}$ = $(\frac{8}{5})^{\frac{-1}{3}} = (\frac{5}{8})^{\frac{1}{3}}$

6. *Find the sum of* $\frac{3}{4} + \frac{3.5}{4.8} - \frac{3.5.7}{4.8.12} + ...$

Solution

S = $\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+......$

-S = $-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+......$

-S + 1 = $1-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+......$

$=(1+x)^{\frac{-p}{q}}$ where p = 3, q = 2: $\frac{x}{q}=\frac{1}{4}\Rightarrow x=\frac{2}{4}=\frac{1}{2}$

-S + 1 = $(1+\frac{1}{2})^{\frac{-3}{2}} = (\frac{3}{2})^{\frac{-3}{2}} = (\frac{2}{3})^{\frac{3}{2}}$

$1-(\frac{2}{3})^{\frac{3}{2}}$ = S

7. *Sum the series to* inf*inity* $\frac{1.4}{5.10}-\frac{1.4.7}{5.10.15}+\frac{1.4.7.10}{5.10.15.20}-....$

Solution

The numerators form an A.P. with 3 as common difference and the denominators are factorials, each of whose factors has been multiplied by 5.
The series can be written as

$$S=\frac{\frac{1}{3}.\frac{4}{3}}{1.2}(\frac{-3}{5})^2+\frac{\frac{1}{3}.\frac{4}{3}.\frac{7}{3}}{1.2.3}(\frac{-3}{5})^3+\frac{\frac{1}{3}.\frac{4}{3}.\frac{7}{3}\frac{10}{3}}{1.2.3.4}(\frac{-3}{5})^4+.......$$

Put n = $\frac{1}{3}$ and x = - $\frac{3}{5}$

S = $\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+.....+\frac{n(n+1)(n+2).....(n+(r-1))}{1.2.3.....r}x^r+\ldots\infty$

= 1+ nx + $\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+\ldots+\frac{n(n+1)(n+2).....(n+(r-1))}{1.2.3.....r}x^r+\ldots$-(1+ nx)
= $(1-x)^{-n}$ - 1 - nx

$=(1+\frac{3}{5})^{-\frac{1}{3}}-1+\frac{1}{3}.\frac{3}{5}$ $=\frac{1}{2}(5)^{\frac{1}{3}}-\frac{4}{5}$

8. *Sum the series to* inf*inity* $\frac{15}{16}-\frac{15.21}{16.24}+\frac{15.21.27}{16.24.32}+....$

Solution

The numerators form an A.P. with 3 as common difference and the denominators are factorials, each of whose factors has been multiplied by 5.

The series can be written as

$$S = \frac{\frac{15}{6}}{2}(\frac{6}{8}) + \frac{\frac{15}{6}.\frac{21}{6}}{2.3}(\frac{6}{8})^2 + + \frac{\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{2.3.4}(\frac{6}{8})^3 + ...$$

Multiply $\frac{9}{6}$ on both sides

$$\frac{9}{6}S = \frac{\frac{9}{6}.\frac{15}{6}}{1.2}(\frac{6}{8}) + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{1.2.3}(\frac{6}{8})^2 + + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{1.2.3.4}(\frac{6}{8})^3 + ...$$

Multiply $(\frac{6}{8})$ on both sides

$$\frac{9}{6}.(\frac{6}{8})S = \frac{\frac{9}{6}.\frac{15}{6}}{1.2}(\frac{6}{8})^2 + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}}{1.2.3}(\frac{6}{8})^3 + + \frac{\frac{9}{6}.\frac{15}{6}.\frac{21}{6}.\frac{27}{6}}{1.2.3.4}(\frac{6}{8})^4 + ...$$

Put n = $\frac{9}{6}$ and x = $\frac{6}{8}$

$$\frac{9}{8}S = \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \ldots.. + \frac{n(n+1)(n+2).....(n+(r-1))}{1.2.3.....r}x^r + \ldots \infty$$

$$= 1 + nx + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \ldots + \frac{n(n+1)(n+2).....(n+(r-1))}{1.2.3.....r}x^r + \ldots -(1+nx)$$

$= (1 - x)^{-n} - 1 - nx$

$$= (1 - \frac{6}{8})^{-\frac{3}{6}} - (1 + \frac{9}{6}.\frac{6}{8}) \qquad = (\frac{1}{4})^{-\frac{3}{2}} - (1 + \frac{9}{8})$$

$$\frac{9}{8}S = \frac{47}{8}$$

$$S = \frac{47}{9}$$

2.3.14.2 PROBLEMS FOR PRACTICE

1. Prove that $\left(1+x\right)^n = 2^n\{1 - n\frac{1+x}{1+x} + \frac{n(n+1)}{1.2}(\frac{1-x}{1+x})^2 - .....\}$
2. Show that $x^n = 1 + n(1 - \frac{1}{x}) + \frac{n(n+1)}{1.2}(1 - \frac{1}{x})^2 + ...$

3. Find the sum to infinity of the following series

(1) $\frac{2}{6}+\frac{2.5}{6.12}-\frac{2.5.8}{6.12.18}+......$

(2) $1+\frac{2}{1!}\frac{1}{6}+\frac{2.5}{2!}\frac{1}{6^2}-\frac{2.5.8}{3!}\frac{1}{6^3}+......$

(3) $\frac{1}{5}-\frac{1.4}{5.10}+\frac{1.4.7}{5.10.15}-......$

(4) $\frac{4}{20}+\frac{4.7}{20.30}-\frac{4.7.10}{20.30.40}+......$

(5) $\frac{5}{3.6}\frac{1}{4^2}+\frac{5.8}{3.6.9}\frac{1}{4^3}+\frac{5.8.11}{3.6.9.12}\frac{1}{4^4}+......$

2.3.15 FIND THE APPROXIMATE VALUES

1. *Find correct to six places of decimals the value of* $(\frac{1}{(9998)})^{\frac{1}{4}}$

Solution

$$(\frac{1}{(9998)})^{\frac{1}{4}}=\frac{1}{(9998)^{\frac{1}{4}}}=\frac{1}{(10000-2)^{\frac{1}{4}}}$$

$$=\frac{1}{(10^4-2)^{\frac{1}{4}}}=\frac{1}{10(1-\frac{2}{10^4})^{\frac{1}{4}}}=\frac{(1-\frac{2}{10^4})^{\frac{-1}{4}}}{10}$$

2. *Calculate correct to six places of decimals* $(1.01)^{\frac{1}{2}}-(0.99)^{\frac{1}{2}}$.

Solution

Write x = 0.01

$$(1.01)^{\frac{1}{2}}=(1+x)^{\frac{1}{2}}$$

$$=1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{-1}{2})}{2!}x^2+\frac{\frac{1}{2}(\frac{-1}{2})(-\frac{3}{2})}{3!}x^3+......$$

$$(0.99)^{\frac{1}{2}}=(1-x)^{\frac{1}{2}}$$

$$=1-\frac{1}{2}x+\frac{\frac{1}{2}(\frac{-1}{2})}{2!}x^2-\frac{\frac{1}{2}(\frac{-1}{2})(-\frac{3}{2})}{3!}x^3+......$$

$$(1.01)^{\frac{1}{2}}-(0.99)^{\frac{1}{2}}=2\left\{\frac{1}{2}x+\frac{\frac{1}{2}(\frac{-1}{2})(-\frac{3}{2})}{3!}x^3+\frac{\frac{1}{2}(\frac{-1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{5!}x^5+......\right.$$

$= 2\left\{\frac{1}{2}x + \frac{1}{16}x^3 + \frac{7}{256}x^5 + ...\right\}$

$= x + \frac{1}{8}x^3 + \frac{7}{128}x^5 + ...$

$= (0.01) + \frac{1}{8}(0.01)^3 + \frac{7}{128}(0.01)^5 + ...$

$= (0.01) + \frac{1}{8}(0.000001) +$ terms not affecting the 8^{th} decimal place

$= (0.01) + (0.000000125)$

$= 0.010000125$

$(1.01)^{\frac{1}{2}} - (0.99)^{\frac{1}{2}} = 0,010000$ correct to six places of decimals.

2.3.15.1 PROBLEMS FOR PRACTICE

Find the value of

1. $(998)^{1/3}$
2. $(1.02)^{\frac{3}{2}} - (0.98)^{\frac{3}{2}}$
3. $(1003)^{\frac{1}{3}} - (997)^{\frac{1}{3}}$
4. $(3.96)^{1/2}$

2.4 EXPONENTIAL SERIES

$1. e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^r}{r!} + ...\infty$

$2. e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ... + (-1)^r \frac{x^r}{r!} + ...\infty$

$3. \frac{e^x + e^{-x}}{2} = \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\infty$

$4. \frac{e^x - e^{-x}}{2} = \sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\infty$

$1. e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$ (e = 2.718)(2< e < 3)

$2. e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...\infty$

$2. \frac{e^1 + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + ...\infty$

$4. \frac{e^1 - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + ...\infty$

2.5 LOGARITHMIC SERIES

When -1 < x < 1 (or) |x| < 1

1. $log\ (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$...............∞

2. $log\ (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} -$...............∞

3. - $log\ (1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} +$...............∞

4. $\frac{1}{2}\ log\ \left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} +$...............∞

5. $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +$...............∞

2.5.1 NOTE

1. $e^1 - 1 = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} +$.................∞

2. $e^1 - 2 = \frac{1}{2!} + \frac{1}{3!} +$.................∞

3. $e^1 - \frac{5}{2} = \frac{1}{3!} + \frac{1}{4!} +$.................∞

4. $\log 2 - 1 = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} +$...............∞

5. $\log 2 - \frac{1}{2} = \frac{1}{3} - \frac{1}{4} + \frac{1}{5} -$...............∞

1. *What do e and e^x stand for? Prove that e lies between 2 and 3 and that it is irrational (incommensurable).*

Solution

$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and e^{xn} $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$

we know from Binomial Theorem that if n be less than 1,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + ...$$

$$= 1 + 1 + \frac{(1-\frac{1}{n})}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + ... \quad \text{------------}(1)$$

The above relation holds good for all values of n. choosing n very large so that products of the form $(1-\frac{1}{n})$ $(1-\frac{2}{n})$ etc. all tend to 1 as $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}$ will tend to zero. Hence taking limit of both sides of (1) when n is infinite, we get n $= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty$

$$=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty$$

Similarly proceeding as above, we can prove that

$e^{x} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$ and also $= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$

$$=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+......\infty \quad \text{-----------(2)}$$

e lies between 2 and 3

e $=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty$ -------------(3)

clearly e is greater than 2 and since,

$\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty \qquad \frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+......\infty$

Is less than $\frac{1}{2}+\frac{1}{2^{2}}+\frac{1}{2^{3}}+......\infty$

Or less than $\dfrac{\frac{1}{2}}{1-\frac{1}{2}}$ (summing an infinite G.P) or < 1, hence e is less than

$1+\frac{1}{1!}+1$, i.e, less than 3 from (3).

Therefore e is greater than 2 but less than 3.

e is irrational

If possible suppose e = p\q, where p and q are +ve integers; then

$$e=\frac{p}{q}=(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....+\frac{1}{q!})+\frac{1}{(q+1)!}+\frac{1}{(q+2)!}+....$$

Now multiplying both sides by q!, we get

$$\frac{p}{q}.q!=(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....+\frac{1}{q!})q!+\frac{q!}{(q+1)!}+\frac{1}{(q+2)(q+1)q!}+.......$$

or p.(q-1)! = an integer + $\frac{q!}{(q+1)!}+\frac{1}{(q+2)(q+1)q!}+.......$ -----------(4)

Now $\frac{1}{(q+1)}+\frac{1}{(q+2)(q+1)}+.......$ is less than $\frac{1}{(q+1)}+\frac{1}{(q+1)^2}+.......$

or less than $\frac{\frac{1}{q+1}}{1-\frac{1}{q+1}}$ [summing an infinite G.P] or less than $\frac{1}{q}$.

Also $\frac{1}{q+1}+\frac{1}{(q+2)(q+1)}+.....$ is clearly greater than $\frac{1}{q+1}$. Thus $\frac{1}{q+1}+\frac{1}{(q+2)(q+1)}+.....$ is fraction whose sum lies between $\frac{1}{q}$ and $\frac{1}{q+1}$ and is thus a proper fraction.

Hence from (4), we get

An integer = an integer + fraction, which is clearly absurd. Therefore e is irrational.

2.5.2 PROBLEMS FOR PRACTICE

1. *Find the sum to infinity of the series* $1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+......\infty$

Solution

$$T_n=\frac{n^3}{n!}=\frac{n^2.n}{(n-1)!.n}=\frac{n^2}{(n-1)!}=\frac{(n-1).(n-2)+3n-2}{(n-1)!}$$

$$T_n=\frac{1}{(n-3)!}+\frac{3n-3+1}{(n-1)!}=\frac{1}{(n-3)!}+\frac{3(n-1)+1}{(n-1)!}$$

$$=\frac{1}{(n-3)!}+\frac{3(n-1)}{(n-1)(n-2)!}+\frac{1}{(n-1)!}$$

$$=\frac{1}{(n-3)!}+3.\frac{1}{(n-2)!}+\frac{1}{(n-1)!}$$

Putting n = 1,2,3,4 and remembering that $\frac{1}{(-n)!}=0$ and 0! = 1

$T_1 = 0 + 3.0 + 1$

$T_2 = 0 + 3.1 + \frac{1}{1!}$

$T_3 = 1 + 3.\frac{1}{1!} + \frac{1}{2!}$

$T_4 = \frac{1}{1!} + 3.\frac{1}{2!} + \frac{1}{3!}$ and so on.

Adding vertically, we get

$$S_\infty=(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty)+3(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty)+(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty)$$

$=e + 3e + e = 5e.$

2. *Show that* $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+...... = e^{-1}$

Solution

$T_n = \frac{2n}{(2n+1)!} = \frac{2n+1-1}{(2n+1)!} = \frac{(2n+1)-1}{(2n+1)!}$

$= \frac{(2n+1)}{(2n+1)2n!} - \frac{1}{(2n+1)!}$

$= \frac{1}{2n!} - \frac{1}{(2n+1)!}$

Putting n = 1,2,3,4 and remembering that $\frac{1}{(-n)!} = 0$ and 0! = 1

$T_1 = \frac{1}{2!} - \frac{1}{3!}$

$T_2 = \frac{1}{4!} - \frac{1}{5!}$ and so on.

Adding vertically, we get

$S_\infty = (\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ......\infty) = e^{-1}$

3. *Show that* $3 + \frac{5}{1!} + \frac{7}{2!} + \frac{9}{3!} + ... = 5e$

Solution

$T_n = \frac{2n+1}{(n-1)!} = \frac{2(n-1)+3}{(n-1)!}$

$= \frac{2}{(n-2)!} + \frac{3}{(n-1)!}$

Putting n = 1,2,3,4 and remembering that $\frac{1}{(-n)!} = 0$ and 0! = 1

Adding vertically, we get

$S_\infty = 5(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\infty) = 5e$

4. Prove that $(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...\infty)^2 = 1 + (1 + \frac{1}{3!} + \frac{1}{5!} + ...\infty)^2$

Solution

$$L.H.S=\left(\frac{e+e^{-1}}{2}\right)^2 \;=\frac{e^2+e^{-2}+2e^1.e^{-1}}{4} \;=\;=\frac{e^2+e^{-2}+2}{4}$$

$$R.H.S=\left(\frac{e-e^{-1}}{2}\right)^2 \;=\frac{4+(e^2+e^{-2}-2e^1.e^{-1})}{4} \;=\frac{e^2+e^{-2}+2}{4}=L.H.S$$

5. *Sum the series* $\frac{1}{1!}+\frac{1+3}{2!}x+\frac{1+3+5}{3!}x^2+\frac{1+3+5+7}{4!}x^3+......\infty$

Solution

$$T_n=\frac{1+3+5+7+....n\ terms}{n!}x^{n-1}$$
$$=\frac{\frac{n}{2}[2.1+(n-1)2]}{n!}x^{n-1}=\frac{n^2}{n!}x^{n-1}$$
$$=\frac{n}{(n-1)!}x^{n-1}=\frac{n-1+1}{(n-1)!}x^{n-1}$$

Putting n = 1,2,3,4 and remembering that $\frac{1}{(-n)!}=0$ and 0! = 1

$$T_1=0+1$$
$$T_2=x+\frac{x}{1!}$$
$$T_3=\frac{x^2}{1!}+\frac{x^2}{2!},$$
$$T_4=\frac{x^3}{2!}+\frac{x^3}{3!},\text{and so on.}$$

Adding vertically, we get

$$S_\infty=(x+\frac{x^2}{1!}+\frac{x^3}{2!}+......\infty)+(1+\frac{x}{1!}+\frac{x^2}{2!}+......\infty)$$
$$=x(1+\frac{x}{1!}+\frac{x^2}{2!}+......\infty)+(1+\frac{x}{1!}+\frac{x^2}{2!}+......\infty)$$
$$=xe^x+e^x=(x+1)e^x$$

6. *Show that* $1+\frac{1+a}{2!}+\frac{1+a+a^2}{3!}+\frac{1+a+a^2+a^3}{4!}+.....=\frac{e^a-e}{a-1}$

Solution

$$T_n=\frac{1+a+a^2+a^3+....n\ terms}{n!}$$
$$=\frac{1.(a^n-1)}{(a-1)n!}=\frac{1}{(a-1)}\left[\frac{a^n}{n!}-\frac{1}{n!}\right]$$

Putting n = 1,2,3,4 and remembering that

$$T_1 = \frac{1}{(a-1)}\left[\frac{a^n}{1!} - \frac{1}{1!}\right],$$

$$T_2 = \frac{1}{(a-1)}\left[\frac{a^2}{2!} - \frac{1}{2!}\right],$$

$$T_3 = \frac{1}{(a-1)}\left[\frac{a^3}{3!} - \frac{1}{3!}\right], \text{ and so on.}$$

Adding vertically, we get

$$S_\infty = \frac{1}{(a-1)}\left[\{\frac{a}{1!} + \frac{a^2}{2!} + \frac{a^3}{3!} + ...\} - \{\frac{1}{1!} + \frac{1}{2!} + ...\infty\}\right]$$

$$= \frac{1}{(a-1)}\left[(e^a - 1) - (e^1 - 1)\right]$$

$$= \frac{e^a - e}{a-1}$$

2.5.3 NOTE : If we put a = 2, we get,

$$1 + \frac{1+2}{2!} + \frac{1+2+2^2}{3!} + \frac{1+2+2^2+2^3}{4!} + ... = e^2 - e$$

7. Pr*ove that* $1 + \frac{2^4}{2!} + \frac{3^4}{3!} + \frac{4^4}{4!} + ...\infty = 15e$

Solution

$$T_n = \frac{n^4}{n!} = \frac{n^3.n}{(n-1)!.n} = \frac{n^3}{(n-1)!} = \frac{(n-1).(n-2)(n-3) + 6n^2 - 11n + 6}{(n-1)!}$$

$$T_n = \frac{1}{(n-4)!} + \frac{6n^2 - 11n + 6}{(n-1)!} = \frac{1}{(n-4)!} + \frac{6(n-1)(n-2) + 18n - 12 - 11n + 6}{(n-1)!}$$

$$= \frac{1}{(n-4)!} + \frac{6}{(n-3)!} + \frac{7n-6}{(n-3)!}$$

$$= \frac{1}{(n-4)!} + \frac{6}{(n-3)!} + \frac{7(n-1)+7-6}{(n-1)!}$$

$$= \frac{1}{(n-4)!} + \frac{6}{(n-3)!} + \frac{7(n-1)}{(n-1)(n-2)!} + \frac{1}{(n-1)!}$$

Putting n = 1,2,3 and remembering that $\frac{1}{(-n)!} = 0$ and 0! = 1

Adding vertically, we get

$$S_\infty = (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\infty) + 6(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...\infty)$$

$$+7(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty)+(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+......\infty)$$
$$= e + 6e + 7e + e = 15e.$$

8. Prove that $1+\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+......\infty=(x^3+6x^2+7x+1)e^x$

Solution

$$S=\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+......\infty$$

and hence we have to find the value of 1+S.

$$T_n=\frac{(n+1)^3}{n!}x^n$$
$$=(\frac{n^3+3n^2+3n+1}{n!})x^n$$
$$=\frac{n^2+3n+3}{(n-1)!}x^n+\frac{1}{n!}x^n$$

$$=\frac{(n-1).(n-2)+6n+1}{(n-1)!}x^n+\frac{1}{n!}x^n$$
$$T_n=\frac{1}{(n-3)!}x^n+\frac{6(n-1)+7}{(n-1)!}x^n+\frac{1}{n!}x^n$$
$$=x^3\frac{x^{n-3}}{(n-3)!}+\frac{6}{(n-2)!}x^n+\frac{7}{(n-1)!}x^n+\frac{1}{n!}x^n$$

$$=x^3\frac{x^{n-1}}{(n-3)!}+6.x^2\frac{x^{n-2}}{(n-2)!}x^n+7x\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}$$

Putting n = 1,2,3 and remembering that $\frac{1}{(-n)!}=0$ and 0! = 1

Adding vertically, we get

$$S=x^3e^x+6x^2e^x+7x\,e^x+\left(e^x-1\right)$$

$$1+S=\left(x^3+6x^2+7x+1\right)e^x.$$

9. *By comparing two different expansion of* $\left(\frac{e^x-e^{-x}}{2}\right)^n$ *show that*

$$n^n-n\left(n-2\right)^n\frac{n(n-1)}{2!}(n-4)^n-.....+to\ \left(n+1\right)terms=2^n.n!$$

Solution

$$\left(\frac{e^x-e^{-x}}{2}\right)^n=\frac{1}{2^n}\left[e^{nx}-ne^{(n-1)x}.e^x+\frac{n(n-1)}{2!}e^{(n-2)x}.e^{-2x}-....\right]$$

$$=\frac{1}{2^n}\left[e^{nx}-ne^{(n-2)x}+\frac{n(n-1)}{2!}e^{(n-4)x}-....\right]------(A)$$

Now $e^{nx}=1+(nx)+\frac{(nx)^2}{2!}+\frac{(nx)^3}{3!}+........+\frac{(nx)^n}{n!}+.......$

i.e., the coefficient of x^n in the expansion of e^{nx} is $\frac{n^n}{n!}$.

Similarly, the coefficient of x^n in the expansion of $-ne^{(n-2)x}$ is $-\frac{n(n-2)^n}{n!}$

the coefficient of x^n in the expansion of $\frac{n(n-1)}{2!}e^{(n-4)x}$ is $\frac{n(n-1)}{2!},\frac{(n-4)^n}{n!}$ ...and so on.

Hence the coefficient of x^n in (A) is

$$\frac{1}{2^n}\left[\frac{n^n}{n!}+\frac{n(n-2)^n}{n!}+\frac{n(n-1)}{2!}\frac{(n-4)^n}{n!}-......(n+1)\text{ terms}\right]$$

Again $\left(\frac{e^x-e^{-x}}{2}\right)^n=\left[\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+........\right]^n=x^n\left[1+(\frac{x^2}{3!}+\frac{x^4}{5!}+......)\right]^n$

and clearly the coefficient of x^n is unity. Equating the coefficient of x^n obtained from R.H.S. and L.H.S. of (A), we get the required result.

10. By means of the identity $e^{x^2+\frac{1}{x^2}+2}=e^{(x+\frac{1}{x})^2}$ or otherwise, Show that

$$e^2\{1+\frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\frac{1}{(3!)^2}+....\}=1+\frac{2!}{(1!)^2}+\frac{4!}{(2!)^2}+\frac{6!}{(3!)^2}+....$$

Solution

$$e^{x^2+\frac{1}{x^2}+2}=e^{x^2}\,e^{\frac{1}{x^2}}\,e^2$$

$$=e^2\{1+\frac{x^2}{1!}+\frac{(x^2)^2}{2!}+\frac{(x^2)^3}{3!}+....\}\ \{1+\frac{1}{1!}(\frac{1}{x^2})+\frac{1}{2!}(\frac{1}{x^2})^2+\frac{1}{3!}(\frac{1}{x^2})^3+....\}$$

Clearly on multiplication the term independent of x is

$$=e^2\{1+\frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\frac{1}{(3!)^2}+....\}\text{ --------------(A)}$$

Again $e^{(x+\frac{1}{x})^2}=\{1+\frac{(x+\frac{1}{x})^2}{1!}+\frac{(x+\frac{1}{x})^4}{2!}+\frac{(x+\frac{1}{x})^6}{3!}+.....\}$

Now $(x+\frac{1}{x})^{2n}$ contains (2n + 1) terms and its middle

i.e., $(n+1)^{th}$ term will clearly be independent of x.

$$T_{n+1} = 2nC_n x^{2n-n}.\frac{1}{x^n} = 2nC_n = \frac{2n!}{n!(2n-n)!} = \frac{2n!}{(n!)^2}$$

Putting n = 1, 2, 3, …and so on, the term independent of x in the expansion of $(x+\frac{1}{x})^2$, $(x+\frac{1}{x})^4$, $(x+\frac{1}{x})^6$ and so on are respectively $\frac{2!}{(1!)^2},\frac{4!}{(2!)^2},\frac{6!}{(3!)^2}$ and so on.

Hence the term independent of x^n in the expansion of $e^{(x+\frac{1}{x})^2}$ is $\left[1+\frac{2!}{(1!)^2}\frac{1}{1!}+\frac{4!}{(2!)^2}\frac{1}{2!}+\frac{6!}{(3!)^2}\frac{1}{3!}+....\right]----(B)$

Equating (A) and (B), we get the required result.

11. *Find the value of* $\dfrac{1+\frac{1}{3!}+\frac{1}{5!}+....\infty}{1+\frac{1}{2!}+\frac{1}{4!}+....\infty}$

Solution

$$\frac{1+\frac{1}{3!}+\frac{1}{5!}+....\infty}{1+\frac{1}{2!}+\frac{1}{4!}+....\infty} = \frac{(\frac{e-e^{-1}}{2})}{(\frac{e+e^{-1}}{2})} = \frac{e-\frac{1}{e}}{e+\frac{1}{e}} = \frac{e^2-1}{e^2+1}$$

12. *Find the value of* $\lim_{n\to\infty}(1+\frac{x}{n})^n$, *for any rational x.*

Solution

Formula : $\lim_{n\to\infty}(1+\frac{x}{n})^n = e$

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = \lim_{n\to\infty}\left\{\left[1+\frac{1}{\left(\frac{n}{x}\right)}\right]^{\frac{n}{x}}\right\}^x = \lim_{n\to\infty}\left\{\left[1+\frac{1}{p}\right]^p\right\}^x = e^x$$

2.5.4 FIND THE COEFFICIENTS OF nth TERM

1. *Find the coefficient of* x^n *in the expansion of* $\left(\frac{a+bx+cx^2}{e^x}\right)$

Solution

$$\left(\frac{a+bx+cx^2}{e^x}\right)=\left(a+bx+cx^2\right)e^{-x}$$

$$=\left(a+bx+cx^2\right)\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+........+\frac{(-1)^{n-2}x^{n-2}}{(n-2)!}+\frac{(-1)^{n-1}x^{n-1}}{(n-1)!}+\frac{(-1)^n x^n}{(n)!}+.......\right]$$

Sum of the terms of x^n

$$=\left[c\frac{(-1)^{n-2}x^n}{(n-2)!}+b\frac{(-1)^{n-1}x^n}{(n-1)!}+a\frac{(-1)^n x^n}{(n)!}\right]$$

Co efficient of x^n

$$=\left[c\frac{(-1)^{n-2}}{(n-2)!}+b\frac{(-1)^{n-1}}{(n-1)!}+a\frac{(-1)^n}{(n)!}\right]$$

$$=\left[c\frac{(-1)^{n-2}}{(n-2)!}+b\frac{(-1)^{n-1}}{(n-1)!}+a\frac{(-1)^n}{(n)!}\right]$$

$$=a\frac{(-1)^n}{(n)!}-b\frac{(-1)^n n}{n(n-1)!}+c\frac{(-1)^n n(n-1)}{(n-2)!.n(n-1)}$$

$$=\frac{(-1)^n}{(n)!}\left[a-bn+cn(n-1)\right]$$

2. *Find the coefficient of* x^n *in the* exp*ansion of*

$$1+\frac{(a+bx)}{1!}+\frac{(ax+b)^2}{2!}+\frac{(ax+b)^3}{3!}+.......+\frac{(ax+b)^n}{n!}+.....$$

Solution

$$S=1+\frac{(a+bx)}{1!}+\frac{(ax+b)^2}{2!}+\frac{(ax+b)^3}{3!}+.......+\frac{(ax+b)^n}{n!}+.....$$

$$=1+\frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+.......+\frac{y^n}{n!}+.....=e^y=e^{a+bx}=e^a.\ e^{bx}$$

$$=e^a.\left[1+\frac{bx}{1!}+\frac{(bx)^2}{2!}+\frac{(bx)^3}{3!}+.......+\frac{(bx)^n}{n!}+.....\right]$$

Coefficient of x^n = $\dfrac{e^a.b^n}{n!}$

3. *Find the coefficient of* x^n *in the* exp*ansion of*

$$(2x-2)+\frac{(3-2x)^2}{2!}-\frac{(3-2x)^3}{3!}+.......+\frac{(3-2x)^n}{n!}+.....$$

Solution

$$S=(2x-2)+\frac{(3-2x)^2}{2!}-\frac{(3-2x)^3}{3!}+.......+\frac{(3-2x)^n}{n!}+.....$$

$$=1-(3-2x)+\frac{(3-2x)^2}{2!}-\frac{(3-2x)^3}{3!}+.......+\frac{(3-2x)^n}{n!}+.....$$

$$=1-\frac{y}{1!}+\frac{y^2}{2!}-\frac{y^3}{3!}+......where\ y\ =\ (3\ -\ 2x)$$

$$=\ e^{-y}\ =\ e^{-(3-2x)}\ =\ e^{-3}.e^{2x}$$

$$=\ e^{-3}.[1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+......]$$

$$Coefficient\ of\ x^n=\frac{e^{-3}.2^n}{n!}$$

2.5.5 PROBLEMS FOR PRACTICE

1. *Expand* $\left(e^{5x}\ +\ e^{x}\right)/\ e^{3x}$ *in ascending powers of x.* *Answer* $2\left\{1+\frac{(2x)^2}{(2!)}+\frac{(2x)^4}{(4!)}+..\right\}$

2. Prove that $\frac{1^2}{2!}+\frac{2^2}{3!}+\frac{3^2}{4!}+....\infty=e-1$

3. Prove that $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+....=\frac{3e}{2}$

4. *Show that* $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+..=2e$

5. (*a*) $\frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{37}{4!}+\frac{56}{5!}+....=6e-1$.

*H*int.4, 11,22,37,56, ….*are such that the successive differences*,7, 11, 15, 19,….*are in A.P and hence* $T_n=\frac{2n^2+n+1}{n!}$ *etc.*

(*b*) $\frac{2}{(1!)}+\frac{12}{2!}+\frac{28}{3!}+\frac{50}{4!}+\frac{78}{5!}+....$

6. Prove that the coefficient of x^n in the expansion of $(1+\frac{x^2}{1!}+\frac{x^4}{4!}+....)^2$ is $\frac{2^{n-1}}{n!}$

Hint $(1+\frac{x^2}{1!}+\frac{x^4}{4!}+....)^2=\left(\frac{e^x+e^{-x}}{2}\right)^2=\frac{1}{4}(e^{2x}+e^{-2x}+2)$

$$=\frac{1}{2}+\frac{1}{2}\frac{(e^x+e^{-x})}{2}=\frac{1}{2}+\frac{1}{2}\left\{1+\frac{(2x)^2}{(2!)}+\frac{(2x)^4}{(4!)}+..\right\}$$

7. *Show that* $1+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\frac{1^2+2^2+3^2+4^2}{4!}+.....=\frac{7}{6}e$

$H\text{int}.T_n=\frac{1^2+2^2+3^2+4^2+....n^2}{n!}=\frac{n(n+1)(2n+1)}{6.n!}$

8. *Show that the coefficient of* x^n *in the series* $1+\frac{a+bx}{1!}+\frac{(a+bx)^2}{2!}+.....$ *is* $e^a.\frac{b^n}{n!}$.

$(clue: L.H.S=e^{a+bx}=e^a.e^{bx})$.

9. *Find the value of* $\dfrac{1+\frac{1}{3!}+\frac{1}{5!}+....\infty}{\frac{1}{2!}+\frac{1}{4!}+....\infty}$ $Answer=\frac{e+1}{e-1}$

10. *Find the coefficient of* x^n *in the expansion of* $\left(\frac{4e^{3x}-5e^{-x}}{e^x}\right)$

$H\text{int}:\left(\frac{4e^{3x}-5e^{-x}}{e^x}\right)=(4e^{3x}-5e^{-x})e^{-x}=4e^{2x}-5e^{-2x}$

Coefficient of $x^n==\frac{2^n}{n!}[4-5(-1)^n]$

11. *Find the coefficient of* x^n *in the expansion of* $(1+x)e^{1+x}$

$H\text{int}:(1+x)e^{1+x}=(1+x).e.e^x$. *Co efficient of* $x^n=\frac{e(n+1)}{n!}$

2.5.6 LOGARITHMIC SERIES

Note: The base e is understood unless stated otherwise.

1. *Prove that if* $x<1$ *then* $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......$ *Also find an expansion for* $\{\log(1+x)\}^2$

Solution

We know that $a^y=e^{y\log a}$ (or)

$$a^y=1+(y\log a)+\frac{(y\log a)^2}{2!}+\frac{(y\log a)^3}{3!}+......$$

Now put $a=(1+x)$ in both sides

$$(1+x)=1+y\log(1+x)+\frac{y^2}{2!}\{\log(1+x)\}^2+\frac{y^3}{3!}\{\log(1+x)\}^3+......$$

Now as x is less than unity, we have by Binomial Theorem

$$1+yx+\frac{y(y-1)}{2!}x^2+\frac{y(y-1)(y-2)}{3!}x^3+......$$

$$=1+y\log(1+x)+\frac{y^2}{2!}\{\log(1+x)\}^2+\frac{y^3}{3!}\{\log(1+x)\}^3+......$$

Equating the coefficient of y in both sides of above, we get

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-......$$

Again the coefficient of y^2 in R.H.S is $\frac{1}{2!}\{\log(1+x)\}^2$ and the coefficient of y in

$\frac{(y-1)}{2!}x^2+\frac{(y-1)(y-2)}{3!}x^3+......$ and is equal to $\frac{x^2}{1.2}-\frac{1+2}{3!}x^3+......$

$$\frac{1}{2!}\{\log(1+x)\}^2=\frac{x^2}{1.2}-\frac{1+2}{3!}x^3+......$$

$$\{\log(1+x)\}^2=2\{\frac{x^2}{2}-\frac{1+2}{1.2}\frac{x^3}{3}+\frac{1.2+2.3+3.1}{1.2.3}\frac{x^4}{4}......\}$$

2.5.7 PROBLEMS FOR PRACTICE

1. Prove that $2\log n-\log(n+1)-\log(n-1)=\frac{1}{n^2}+\frac{1}{2n^4}+\frac{1}{3n^6}+........$

Solution

Put $\frac{1}{n}=x$; R.H.S $=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+..$

$$=-\log(1-x)=-\log(1-\frac{1}{n^2})=\log(\frac{n^2}{n^2-1})$$

$$=\log n^2-\log(n-1)(n+1)=2\log n-\log(n-1)-\log(n+1)$$

2. Prove that

$$2logn-log(n-1)-log(n+1)=2\left\{\frac{1}{(2n^2-1)}+\frac{1}{3(2n^2-1)^3}+\frac{1}{5(2n^2-1)^5}+.........\right\}$$

Solution

Put $\dfrac{1}{(2n^2-1)} = x$; R.H.S = $2\{x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+..\}$ = $\log \dfrac{1+x}{1-x}$ = $\log = \dfrac{1+\dfrac{1}{2n^2-1}}{1-\dfrac{1}{2n^2-1}}$

$$= \log\frac{2n^2}{2n^2-2} = \log\frac{n^2}{n^2-1} = \log n^2 - \log(n-1)(n+1)$$
$$= 2\log n - \log(n-1) - \log(n+1)$$

3. $\Pr ove\ that\ \ \dfrac{1}{(n+1)}+\dfrac{1}{2(n+1)^2}+\dfrac{1}{3(n+1)^3}+......=\dfrac{1}{n}-\dfrac{1}{2n^2}+\dfrac{1}{3n^3}-....$

Solution

Putting x = $\dfrac{1}{(n+1)}$, L.H.S = $x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+..$ = - log (1-x) $=-log(1-\dfrac{1}{n+1})$

$$=-log(\frac{n}{n+1}) = \log\frac{n+1}{n} = \log(1+\frac{1}{n})$$
$$=\frac{1}{n}-\frac{1}{2}\left(\frac{1}{n}\right)^2+\frac{1}{3}\left(\frac{1}{n}\right)^3-......$$

$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-......$$

4. *Find the value of* $2\left\{1+\dfrac{(\log_e n)^2}{2!}+\dfrac{(\log_e n)^4}{4!}+...\right\}$

Solution

S = $2\left\{1+\dfrac{(\log_e n)^2}{2!}+\dfrac{(\log_e n)^4}{4!}+...\right\}$

$$=2\left\{1+\frac{(x)^2}{2!}+\frac{(x)^4}{4!}+...\right\}$$

$$2\left\{\frac{e^x+e^{-x}}{2}\right\} = e^{\log_e n}+e^{-\log_e n} = e^{\log_e n}+e^{\log_e \frac{1}{n}} = n+\frac{1}{n}$$

5. *Find the value of* $2\left\{\log 2+\dfrac{(\log_e 2)^3}{3!}+\dfrac{(\log_e 2)^5}{5!}+...\right\}$

Solution

$$S = 2\left\{\log 2 + \frac{(\log_e 2)^3}{3!} + \frac{(\log_e 2)^5}{5!} + ...\right\}$$

$$= 2\left\{x + \frac{(x)^3}{3!} + \frac{(x)^5}{5!} + ...\right\}$$

$$2\left\{\frac{e^x - e^{-x}}{2}\right\} = e^{\log_e 2} - e^{-\log_e 2} = e^{\log_e 2} - e^{\log_e \frac{1}{2}} = 2 - \frac{1}{2} = \frac{3}{2}$$

6. *Find the sum of* $\left\{\frac{3}{1!} + \frac{5}{3!} + \frac{7}{5!} + ...\right\}$

$$S = \left\{\frac{3}{1!} + \frac{5}{3!} + \frac{7}{5!} + ...\right\}$$

$$T_n = \frac{2n+1}{(2n-1)!}$$

$$\frac{2n+1}{(2n-1)!} = \frac{A}{(2n-1)!} + \frac{B}{(2n-2)!}$$

$2n + 1 = A + B(2n-1)$

$2B = 2$ A-B =1

$B = 1$ A = 2

$$\mathrm{T}_n = \frac{2}{(2n-1)!} + \frac{1}{(2n-2)!}$$

$$S = \sum_{n=1}^{\infty} T_n$$

$$T_1 = \frac{2}{1!} + \frac{1}{1!}$$

$$T_2 = \frac{2}{3!} + \frac{1}{2!}$$

$$T_3 = \frac{2}{5!} + \frac{1}{4!}$$

....

....

$$\sum_{n=1}^{\infty} T_n = 2\left\{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + ...\right\} + \left\{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...\right\}$$

$$S = 2\left\{\frac{e^1 - e^{-1}}{2}\right\} + \frac{e + e^{-1}}{2} = e + \frac{e}{2} - e^{-1} + \frac{e^{-1}}{2} = \frac{3e}{2} - \frac{e^{-1}}{2} = \frac{1}{2}(3e - \frac{1}{e})$$

7. *Find the sum of the* series $\{\log_2 e - \log_4 e + \log_8 e - ...\}$

Solution

$$S = \{\log_2 e - \log_4 e + \log_8 e - ...\}$$

$$= \left\{\frac{1}{\log_e 2} - \frac{1}{\log_e 4} + \frac{1}{\log_e 8} - ...\right\}$$

$$= \left\{\frac{1}{\log_e 2} - \frac{1}{2\log_e 2} + \frac{1}{3\log_e 2} - ...\right\}$$

$$= \frac{1}{\log_e 2}\left\{1 - \frac{1}{2} + \frac{1}{3} - ...\right\}$$

$$= \frac{1}{\log_e 2}, \log_e 2 = 1$$

8. $If\ y = 1\ + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...then\ find\ the\ value\ of\ \ x$

Solution

$$y = 1\ + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$y = e^x$$

$$\log y = \log e^x$$

$$\log y = x$$

9. *Find the value of* n *and* x *in sum* min *g up the series* $\left\{\frac{15}{16} + \frac{15.21}{16.24} + ...\infty\right\}$

Solution

$$S \quad = \left\{\frac{15}{16} + \frac{15.21}{16.24} + ...\infty\right\}$$

$$= \left\{\frac{15}{2.8} + \frac{15.21}{2.8.3.8} + ...\infty\right\}$$

$$=\left\{\frac{15}{2!}(\frac{1}{8})+\frac{15.21}{3!}(\frac{1}{8})^2+\frac{15.21.27}{4!}(\frac{1}{8})^3.....\infty\right\}$$

$$\frac{9S}{8}=\left\{\frac{9.15}{2!}(\frac{1}{8})^2+\frac{9.15.21}{3!}(\frac{1}{8})^3+\frac{9.15.21.27}{4!}(\frac{1}{8})^4.....\infty\right\}$$

Here P = 9, q = 6, $\frac{x}{q}=\frac{1}{8}$ => $x=\frac{3}{4}$,

$n=\frac{3}{2}$ 10. *Find the sum of the series* $1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{1.2}\left(1-\frac{1}{x}\right)^2+....$

Solution

$$S = 1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{1.2}\left(1-\frac{1}{x}\right)^2+....$$

$$= 1 + ny + \frac{n(n+1)}{2!}y^2+......=(1-y)^{-n}=\left[1-(1-\frac{1}{x})\right]^{-n}=x^n$$

11. *If* x *is numerically less than unity, prove that*

$$\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\frac{4}{5}x^5+.....=\frac{x}{1-x}+\log(1-x)$$

Solution

$$L.H.S=(1-\frac{1}{2})x^2+(1-\frac{1}{3})x^3+(1-\frac{1}{4})x^4+......=(x^2+x^3+x^4+x^5+...)-(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+...)$$

Adding x in both the brackets, we get

$$=(x+x^2+x^3+x^4+...)-(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+...)$$

$$=\frac{x}{1-x}+\log(1-x)$$

12. *If* $\log\left(1-x+x^2\right)$ *be expanded in ascending powers of* x *in the form* $a_1x + a_2x^2+a_3x^3+.....$ *show that* $a_3+a_6+a_9+.....=\frac{2}{3}\log 2.$

Solution

$$\log\left(1-x+x^2\right)=\log\frac{1+x^3}{1+x}=\log(1+x^3)-\log(1+x)$$

$$= \{x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 - \frac{1}{4}x^{12} + ...+(-1)^{n-1}\frac{x^{3n}}{n} + ...\}$$

$$-\{x - \frac{x^2}{2} + \frac{x^3}{3} - ......+(-1)^{n-1}\frac{x^n}{n} + ...\}$$

$$= \{a_1 x + a_2 x^2 + a_3 x^3 + ...\} given$$

Hence a_n is the coefficient of x^n. Also a_{3n} is the coefficient of x^{3n} in the above expansion and is equal to

$$\frac{(-1)^{n-1}}{n} - (-1)^{3n-1}\frac{1}{3n} = \frac{(-1)^{n-1}}{n} - \frac{(-1)^{2n}.(-1)^{n-1}}{3n} \quad \text{(or)}$$

$$a_{3n} - (-1)^{n-1}\left(\frac{1}{n} - \frac{1}{3n}\right) = (-1)^{n-1}.\frac{2}{3n} - - - - - - - - (1)$$

Putting n = 1,2,3…..in (1) we get

$a_3 = \frac{2}{3}$, $a_6 = -\frac{2}{3}, \frac{1}{2}$, $a_9 = \frac{2}{3}, \frac{1}{4}$…..

$$= \{a_3 + a_6 + a_9 + ...\} = \frac{2}{3}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...) = \frac{2}{3}\log(1+1) = \frac{2}{3}\log 2.$$

2.5.8 PROBLEMS FOR PRACTICE

1. *Find the value of* $\left\{\frac{\log 2}{1!} - \frac{(\log_e 2)^2}{2!} + \frac{(\log_e 2)^3}{3!} - ...\right\}$ Answer S = ½

2. *Find the sum of* $\left\{\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + ...\right\}$ Answer S = 1/e

3. *Find the sum of* $\left\{\frac{2^2}{1!} + \frac{2^4}{3!} + \frac{2^6}{5!} + ...\right\}$ Answer S = $\frac{e^4 - 1}{e^2}$

.

4. *Find the sum of* $\left\{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...\right\}^2 - \left\{1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + ...\right\}^2$ Answer : 1

5. *Find the sum of the* series $\left\{1 + \frac{6}{3!} + \frac{11}{5!} + \frac{16}{7!} + ...\right\}$

$H\text{int}: T_n = \frac{5n-4}{(2n-1)!}$ *Answer*: $\frac{e}{2} + \frac{2}{e}$

6. *Find the sum of the* series $\left\{1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+...\right\}$

$Hint: T_n = \frac{2n-1}{(n)!}$ $Answer: e+1$

7. *Find the sum of the* series $\{\log_3 e - \log_9 e + \log_{27} e - ...\}$

$Answer: \frac{\log_e 2}{\log_e 3}$

8. *Find the sum of* $\left\{\frac{2n}{(n^2+1)}+\frac{1}{3}(\frac{2n}{(n^2+1)})^3+\frac{1}{5}(\frac{2n}{(n^2+1)})^5+.........\right\}$

$Hint: S = x+\frac{x^3}{3}+\frac{x^5}{5}+......$ $Answer: \log(\frac{n+1}{n-1})$

9. *Find the sum of* $\left\{\frac{1}{(2n+1)}+\frac{1}{3}(\frac{1}{(2n+1)})^3+\frac{1}{5}(\frac{1}{(2n+1)})^5+.........\right\}$

$Hint: S = x+\frac{x^3}{3}+\frac{x^5}{5}+......$ $Answer: \log\sqrt{\frac{n+1}{n}}$

10. *Find the sum of* $\left\{\frac{a-1}{(a+1)}+\frac{1}{3}(\frac{a-1}{(a+1)})^3+\frac{1}{5}(\frac{a-1}{(a+1)})^5+.........\right\}$

$Hint: S = x+\frac{x^3}{3}+\frac{x^5}{5}+......$ $Answer: \frac{1}{2}\log a$

11. *Find the sum of* $\left\{\frac{1}{(n+1)}+\frac{1}{2}(\frac{1}{(n+1)})^2+\frac{1}{3}(\frac{1}{(n+1)})^3+.........\right\}$

$Hint: S = x+\frac{x^2}{2}+\frac{x^3}{3}+......$ $Answer: \log(1+\frac{1}{n})$

12. *Find the sum of* $\left\{\frac{1}{e}+\frac{1}{2}\frac{1}{e^2}+\frac{1}{3}\frac{1}{e^3}+.........\right\}$

$Hint: S = x-\frac{x^2}{2}+\frac{x^3}{3}+......$ $Answer: \log(1+\frac{1}{e})$

13. *Find the sum of* $\left\{\frac{1}{7}+\frac{1}{3}(\frac{1}{7})^3+\frac{1}{5}(\frac{1}{7})^5+.........\right\}$

$Hint: S = x+\frac{x^3}{3}+\frac{x^5}{5}+......$ $Answer: \log\sqrt{\frac{4}{3}}$

14. *Find the sum of* $\left\{\frac{1}{3}-\frac{1}{2}(\frac{1}{3})^2+\frac{1}{3}(\frac{1}{3})^3+.........\right\}$

$Hint: S = x-\frac{x^2}{2}+\frac{x^3}{3}+......Answer: \log\frac{4}{3}$

15. *Find the sum of* $\left\{3\log\ 2+\frac{1}{4}-\frac{1}{2}(\frac{1}{4})^2+\frac{1}{3}(\frac{1}{4})^3+.........\right\}$

$Hint: S = \log 2^3+x-\frac{x^2}{2}+\frac{x^3}{3}+......Answer: \log 10$

16. *Find the sum of* $\left\{1+x\log\ _e a+\frac{(x\log\ _e a)^2}{2!}+..........+\frac{(x\log\ _e a)^r}{r!}+........\right\}$

$Hint: S = 1+x+\frac{x^2}{2!}+......+\frac{x^r}{r!}+......Answer: a^x$

17. *Find the sum* $\lim_{x\to 0}\frac{e^x-1-\log(1+x)}{x^2}$

. Hint : using L'Hospital rule: $\lim_{x\to 0}\frac{e^x-0-\log(\frac{1}{1+x})}{2x}$

18. Pr*ovethat* $\lim_{x\to 0}\left(\frac{(1+x)}{(1-x)}\right)^{\frac{3}{x}}=6$

Hint : *let* $A = \lim_{x\to 0}\left(\frac{(1+x)}{(1-x)}\right)^{\frac{3}{x}}$

Log A $= \lim_{x\to 0}\frac{3}{x}\log(\frac{1+x}{1-x})$

19. *If* $y=x-\frac{x^2}{2}+\frac{x^3}{3}-.....$ *then find the series of x.*

Hint : $y=\log\left(1+x\right) \Rightarrow e^y=1+x \Rightarrow x=-1+e^y$

Chapter 3

THEORY OF EQUATIONS

Contents

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An equation of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ where $a_0, a_1, a_2, ..., a_{n-1}, a_n$ are all constants and $a_0 \neq 0$ is known as a rational integral equation of the n[th] degree.

Dividing by a_0 the equation becomes

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$$

This equation is of the form

$$x^n + p_1x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$$

This is known as the standard rational integral equation of the n[th] degree on x .i.e., without loss of generality the coefficient of x^n can always be made unity in an equation.

3.2 RAABE'S TEST

Theorem:

If for a series $\sum a_n$ of positive terms, $\lim_{n\to\infty}\left(1 - \frac{a_{n+1}}{a_n}\right) = \lambda$ exists, then the series converges if $\lambda \succ 1$ and diverges if $\lambda \prec 1$.

Proof

If $\lim_{n\to\infty}\left(1 - \frac{a_{n+1}}{a_n}\right) = \lambda \succ 1$, choose a real number p such that $\lambda \succ p \succ 1$.

Then there exists an integer N such that $n\left(1-\frac{a_{n+1}}{a_n}\right) \succ p$ for all $n \geq N$

This shows that $\frac{a_{n+1}}{a_n} \prec 1-\frac{p}{n} \prec \left(1-\frac{1}{n}\right)^p = \frac{(n-1)^p}{n^p}$ for all $n \geq N$

Writing $b_n = \frac{1}{(n-1)^p}$ for $n \geq N$, it can be seen that $\sum b_n$ is convergent and $\frac{a_{n+1}}{a_n} \prec \frac{b_{n+1}}{b_n}$ for all $n \geq N$

Hence the series $\sum a_n$ is convergent.

When $\lambda \prec 1$, it can be proved that the series is divergent. However , for $\lambda = 1$, nothing can be said about convergence of the series.

Raabe's test :

1. *If* $\lambda \succ 1$, *the series converges.*
2. *If* $\lambda \prec 1$, *the series diverges.*
3. *If* $\lambda = 1$, *the series may converge or diverge.*

Example 1

Use Rabee's test for the convergence of the series

$$\frac{1}{2}+\frac{1.3}{2.4}-\frac{1.3.5}{2.4.6}+.......+\frac{1.3.5....(2n-1)}{2.4.6.....(2n)}+....$$

Solution

The nth term of the series is $\mathbf{a_n} = \frac{1.3.5....(2n-1)}{2.4.6.....(2n)}$

We have $n\left(1-\frac{a_{n+1}}{a_n}\right) = n\left(1-\frac{(2n+1)}{(2n+2)}\right) = \frac{n}{2n+2}$

Therefore $\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\frac{n}{2n+2} = \frac{1}{2} \prec 1.$

Hence by Raabe's test, the given series diverges.

Example 2

Show that Rabee's test for the convergence is successful but the ratio test fails for the series.

$$\frac{2}{7}+\frac{2.5}{7.10}-\frac{2.5.8}{7.10.13}+.......+\frac{2.5.8....(3n-1)}{7.10.13.....(3n+4)}+....$$

Solution

The nth term of the series is $a_n = \dfrac{2.5.8....(3n-1)}{7.10.13.....(3n+4)}$

Let us apply D'Alember's ratio test on the series. It can be seen that

We have $\dfrac{a_{n+1}}{a_n} = \dfrac{3n+2}{3n+7}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{3n+2}{3n+7} = 1$$

Hence the ratio test fails for this series. Now let us apply the Raabe's test.

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\frac{5n}{3n+7} = \frac{5}{3} \succ 1.$$

Hence by Raabe's test, the given series converges.
Raabe's test may be successful, even when the ratio test fails.

PROBLEMS FOR PRACTICE
Use Raabe's test for the convergence of the following series

1. $1+\dfrac{2}{6}+\dfrac{2.5}{6.12}+\dfrac{2.5.8}{6.12.18}+.....\infty$
2. $1+\dfrac{10}{9}+\dfrac{10.16}{9.18}+\dfrac{10.16.22}{9.18.27}+......$
3. $1-\dfrac{1}{4}+\dfrac{1.3}{4.8}+\dfrac{1.3.5}{4.8.12}+......$
4. $1-\dfrac{1}{5}+\dfrac{1.4}{5.10}-\dfrac{1.4.7}{5.10.15}+......$
5. $\dfrac{3}{4}+\dfrac{3.5}{4.8}-\dfrac{3.5.7}{4.8.12}+......$
6. $\dfrac{1.4}{5.10}-\dfrac{1.4.7}{5.10.15}+\dfrac{1.4.7.10}{5.10.15.20}-....$

Solving polynomial equations.

The zeros of a polynomial function $P(x)$ are the same as the roots of the polynomial equation $p(x) = 0$. Remember that one of our main goals in algebra is to keep expanding our knowledge of solving equations. In this section we will learn several facts that are useful in solving polynomial equations.
In solving a polynomial equation by factoring, we find that a factor may occur more than once.

3.2.1 Multiplicity

If the factor $x-c$ occurs n times in the complete factorization of the polynomial $P(x)$, then we say that c is a root of the equation $P(x)= 0$ with multiplicity n.

Example

Discuss the multiplicity of the equation $x^2 - 10x + 25=0$ *is equivalent to* $(x-5)(x-5) = 0$

Solution

The only root to this equation is 5.

Since the factor $x-5$ occurs twice, the multiplicity of the root 5 is 2.

3.2.2 n- Root theorem

If $P(x) = 0$ is a polynomial equation with real or complex coefficients and positive degree n, then counting multiplicities. $P(x) = 0$ has n roots.

PROBLEMS FOR PRACTICE

State the degree of each polynomial equation. Find all of the real and imaginary roots to each equation, stating multiplicity when it is greater than 1

.$a)\, 6x^5 + 24x^3 = 0 \quad b)\, (x - 3)^2 (x+ 4)^5 = 0$

Answer : $a)$ *The roots are* $\pm 2i$ *and* 0. *Since* 0 *is a root with multiplicity* 3, *counting multiplicities there are five roots*.
$b)$ 3 *is a root with multiplicity* 2 *and* -4 *is a root with multiplicity* 5.

3.2.3 Conjugate Pairs Theorem

If $P(x) = 0$ is a polynomial equation with real coefficients and the complex number $a + ib(b \neq 0)$ is a root, then the complex number $a - ib$ is also a root.

Example

Find a polynomial equation with real coefficients that has 2 *and* 1 - *i as roots.*

Solution

Since the polynomial is to have real coefficients, the imaginary roots occur in conjugate pairs. So a polynomial with these two roots actually must have at least three roots: $2, 1 - i,$ and $1 + i$. Since each root of the equation comes from a factor of the polynomial, we can write the following equation:

$$(x - 2)(x - [1 - i])(x - [1 + i]) = 0$$

$$(x - 2)(x^2 - 2x + 2) = 0$$

$$x^3 - 4x^2 + 6x - 4 = 0$$

$$(1 - i)(1 + i) = 1 - i^2 = 2$$

This equation has the required solutions and the smallest degree. Any multiple of this equation would also have the required solutions but would not be as simple.

3.3 REMAINDER THEOREM

If f(x) is a polynomial, then f(a) is the remainder when f(x) is divided by x - a.

Divide the polynomial f(x) by x – a until a remainder is obtained which does not involve x. Let the quotient be Q(x) and remainder R.

Then f(x) = (x - a)Q(x) + R

Substituting x = a in the above equation , we get f(a) = R.

3.2.4 IMPORTANT RESULTS

1. If f(a) = 0, the polynomial f(x) has the factor x- a, i.e., if a be the root of the equation f(x) = 0, then x- a is a factor of the polynomial f(x).
2. If f(a) and f(b) are of different signs, then atleast one root of the equation f(x) = 0 must lie between a and b.
3. If f(a) and f(b) have the same sign, it does not follow that f(x) = 0 has no root between a and b. it is evident that when two points are connected by a curve, the portions of the curve between these points must cut the axis and an even number of times or not at all, when the point are on the same side of the axis.
4. If f(x) = 0 is an equation of odd degree, it has atleast one real root whose sign is opposite to that of the last term.
5. If f(x) = 0 is of even degree and the absolute term is negative, equation has at least one positive root and at least one negative root.
6. Every equation f(x) = 0 of the n^{th} degree has n roots and no more.

Problems

1. *If α be a real root of the cubic equation $x^3+px^2+qx+r=0$, of which the coefficients are real, show that the other two roots of the equation are real, if $p^2 \geq 4q+2p\alpha+3\alpha^2$*

Solution

Since α is a root of the equation, x^3+px^2+qx+r is exactly divisible by $x-\alpha$

Let $x^3+px^2+qx+r \equiv (x-\alpha)\left(x^2+ax+b\right)$

Equating the coefficient of powers of x on both sides, we get

$p=-\alpha+a$

$q=-a\alpha+b$

$r=-b\alpha$

Therefore $a=p+\alpha$ and $b=q+a\alpha=q+\alpha(p+\alpha)=q+p\alpha+\alpha^2$

The other two roots of the equation are the roots of

$x^2+(p+\alpha)x+q+p\alpha+\alpha^2=0$

which are real if $(p+\alpha)^2-4(q+p\alpha+\alpha^2)\geq 0$

$p^2-2p\alpha-4q-3\alpha^2\geq 0$

$p^2\geq 2p\alpha+4q+3\alpha^2$

2. *If $x_1, x_2, \ldots\ldots, x_n$ are the roots of the equation $\left(a_1-x\right)\left(a_2-x\right)\left(a_3-x\right)\ldots\ldots\ldots\ldots(a_n-x)+k=0$, then show that $a_1, a_2, \ldots\ldots, a_n$ are the roots of the equation $\left(x_1-x\right)\left(x_2-x\right)\left(x_3-x\right)\ldots\ldots\left(x_n-x\right)-k=0$.*

Solution

Since $x_1, x_2, \ldots\ldots, x_n$, are the roots of the equation $\left(a_1-x\right)\left(a_2-x\right)\left(a_3-x\right)\ldots\ldots(a_n-x)+k=0$,

we have

$\left(a_1-x\right)\left(a_2-x\right)\left(a_3-x\right)\ldots\ldots(a_n-x)+k=\left(x_1-x\right)\left(x_2-x\right)\left(x_3-x\right)\ldots\ldots\left(x_n-x\right)$

Therefore

$\left(a_1-x\right)\left(a_2-x\right)\left(a_3-x\right)\ldots\ldots(a_n-x)=\left(x_1-x\right)\left(x_2-x\right)\left(x_3-x\right)\ldots\ldots\left(x_n-x\right)-k$

Therefore $a_1, a_2, \ldots\ldots, a_n$ are the roots of

$\left(x_1-x\right)\left(x_2-x\right)\left(x_3-x\right)\ldots\ldots\left(x_n-x\right)-k=0$

3. *Show that if a, b, c are real, the roots of $\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=\frac{3}{x}$ are real.*

Solution

On simplifying $\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=\frac{3}{x}$ we get

$$x(x+b)(x+c)+x(x+c)(x+a)+x(x+a)(x+b)-3(x+a)(x+b)(x+c)=0$$

Let $f(x)$ be the expression on the left – hand side. It can easily be seen that $f(x)$ is a quadratic function of x. Therefore

$$f(-a)=-a(b-a)(c-a)$$
$$f(-b)=-b(c-b)(a-b)$$
$$f(-c)=-c(a-c)(b-c)$$

without loss of generality let us assume that a > b > c and a, b, c are all positive.
Then a – b, b – c, a – c are positive.Therefore

$$f(-a)=-ve$$
$$f(-b)=+ve$$
$$f(-c)=-ve$$

The equation has atleast one real root between - a and - b, and another between - b and - c.
The equation can have only two roots since $f(x)=0$ is a quadratic equation.
Therefore the roots of the equations are real.

PROBLEMS FOR PRACTICE

1. *If $f(x)$ is a rational integer function of x and a, b are unequal, show that the remainder in the division of $f(x)$ by $(x-a)(x-b)$ is* $\frac{(x-a)f(b)-(x-b)f(a)}{b-a}$
2. *If $x^3+3pq+q$ has a factor of the form $x^2-2ax+a^2$, show that $q^2+4p^3=0$*
3. *If px^3+qx+r has factor of the form x^2+ax+1, prove that $p^2=pq+r^2$.*
4. *If px^5+qx^2+r has factor of the form x^2+ax+1, prove that $(p^2-r^2)(p^2-r^2+qr)=p^2q^2$.*

3.4 RELATION BETWEEN ROOTS AND COEFFICIENTS

$\mathbf{a}_0x^n+a_1x^{n-1}+\ldots\ldots+a_n=0$ is a polynomial equation where $a_0, a_1, a_2, \ldots, a_n$ are constants. Let $\alpha_1,\alpha_2,\ldots\ldots\ldots,\alpha_n$ be n- roots of the given equation

CASE 1: If $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p^n = 0$, and let $\alpha_1,\alpha_2,\ldots\ldots\ldots,\alpha_n$ be n- roots then find S_1, S_2, S_3 and S_n

$$S_1=\alpha_1.\alpha_2\ldots\ldots\ldots\alpha_n$$
$$S_2=\alpha_1\alpha_2+\alpha_2\alpha_3+\ldots\ldots\ldots+\alpha_{n-1}\alpha_n=P_2$$
$$S_3=\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_3+\ldots\ldots\ldots+\alpha_{n-2}\alpha_{n-1}\alpha_n=-P_3$$
$$S_4=\alpha_1.\alpha_2.\alpha_3\ldots\ldots\ldots\alpha_n=(-1)^nP_n$$

CASE 2: How to find the relation between roots and coefficients of a quadratic equation
Let us take the quadratic equation of the general form $ax^2+bx+c=0$ where a (≠ 0) is the coefficient of x^2, b the coefficient of x and c, the constant term.

Let α and β be the roots of the equation $ax^2 + bx + c = 0$
Now we are going to find the relations of α and β with a, b and c.
Now $ax^2 + bx + c = 0$
Multiplication both sides by 4a (a ≠ 0) we get

$4a^2x^2 + 4abx + 4ac = 0$

$(2ax)^2 + 2 * 2ax * b + b^2 - b^2 + 4ac = 0$

$(2ax + b)^2 = b^2 - 4ac$

$(2ax + b) = \pm\sqrt{b^2 - 4ac}$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Therefore, the roots are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Let $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

Therefore $\alpha + \beta = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$\alpha + \beta = \dfrac{-2b}{2a}$

$\alpha + \beta = \dfrac{-b}{a}$

$\alpha + \beta = \dfrac{-coefficient\ of\ x}{coefficient\ of\ x^2}$

Again $\alpha\ \beta = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ x $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$\alpha\ \beta = \dfrac{b^2 - (b^2 - 4ac)}{4a^2} = \dfrac{c}{a}$

$\alpha\beta = \dfrac{constant\ term}{coefficient\ of\ x^2}$

therefore $\alpha\beta$ and $\alpha + \beta$ represents the required relations between roots (α and β) and coefficients (a, b and c) of equation $ax^2 + bx + c = 0$.

Example1

If the roots of the equation $7x^2 - 4x - 8 = 0$ be α and β, then find $\alpha + \beta$ and $\alpha\beta$

Solution

Sum of roots = $\alpha + \beta = \dfrac{-coefficient\ of\ x}{coefficient\ of\ x^2} = -\dfrac{-4}{7} = \dfrac{4}{7}$

Product of roots = $\alpha\beta = \dfrac{constant\ term}{coefficient\ of\ x^2} = -\dfrac{8}{7}$

To find the conditions when roots are connected by given relations

Sometimes the relation between roots of a quadratic equation is given and we are asked to find the condition i.e., relation between the coefficients a, b and c of quadratic equation. This is easily done using the formula $\alpha + \beta = \frac{-b}{a}$

and $\alpha\beta = \frac{c}{a}$. This will clear when you go through illustrative examples.

Solved examples to find the relation between roots and coefficients of a quadratic equation:

1.*Without solving the equation*, $5x^2 - 3x + 10 = 0$ *find the sum and the product of the roots.*

Solution:

Let α and β be the roots of the given equation.

Then, $\alpha + \beta = -\frac{-3}{5} = \frac{3}{5}$ *and* $\alpha\beta = \frac{10}{5} = 2$

1.*If* α *and* β *are the roots of the equation* $x^2 - 4x + 2 = 0$, *find the val ue of*

$(i)\ \alpha^2 + \beta^2$

$(ii)\ \alpha^2 - \beta^2$

$(iii)\ \alpha^3 + \beta^3$

$(iv)\ \frac{1}{\alpha} + \frac{1}{\beta}$

Solution:

The given equation is $x^2 - 4x + 2 = 0$ .... (i)

According to the problem, α and β are the roots of the equation (i)

Therefore,

$\alpha + \beta = \frac{-b}{a} = -\frac{-4}{1} = 4$ and $\alpha\beta = \frac{c}{a} = \frac{2}{1} = 2$

(i) Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (4)^2 – 2 * 2 = 16 – 4 = 12$.

(ii) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

Now $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = (4)^2 – 4 * 2 = 16 – 8 = 8$

$\Rightarrow \alpha - \beta = \pm \sqrt{8}$

$\Rightarrow \alpha - \beta = \pm 2\sqrt{2}$

Therefore, $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = 4 * (\pm 2\sqrt{2}) = \pm 8\sqrt{2}$.

$(iii)\ \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = (4)^3 - 3 * 2 * 4 = 64 - 24 = 40.$

$(iv) \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{4}{2} = 2.$

Case 3: If α, β and γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$ then find S_1, S_2, S_3

$$S_1 = \alpha + \beta + \gamma = (-1)^1 \frac{b}{a} = \frac{-b}{a}$$
$$S_2 = \alpha\beta + \beta\gamma + \alpha\gamma = (-1)^2 \frac{c}{a} = \frac{c}{a}$$
$$S_3 = \alpha\beta\gamma = (-1)^3 \frac{d}{a} = \frac{-d}{a}$$

Case 4: If α, β and γ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ then find S_1, S_2, S_3, S_4

$$S_1 = \alpha + \beta + \gamma = (-1)^1 p = -p$$
$$S_2 = \alpha\beta + \beta\gamma + \alpha\gamma = (-1)^2 q = q$$
$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \delta\beta\alpha = (-1)^3 r = -r$$
$$S_4 = \alpha\beta\gamma\delta = (-1)^4 s = s$$

Problems

1. *Solve the equation* $32x^3 - 48x^2 + 22x - 3 = 0$ *given that the roots are in* AP

Solution

Let $\alpha - \beta, \alpha, \alpha + \beta$ be the roots

$$\alpha - \beta + \alpha + \alpha + \beta = \frac{48}{32}$$

$$3\alpha = \frac{48}{32}$$

$$\alpha = \frac{1}{2}$$

$\frac{1}{2}$	32	-48	22	-3
	0	16	-16	3
	32	-32	6	0

$$32x^2 - 32x + 16 = 0$$

$$16x^2 - 16x + 3 = 0$$

$$16x^2 - 4x - 12x + 3 = 0$$

$$4x(4x-1) - 3(4x-1) = 0$$

$$(4x-3)(4x-1) = 0$$

The roots are $x = \frac{1}{2}, \frac{1}{4}$ and $\frac{3}{4}$

2. *Solve the equation* $x^3 - 12x^2 + 39x - 28 = 0$ *given that the roots are in* AP

Solution

Let $\alpha - \beta, \alpha, \alpha + \beta$ be the roots

$$\alpha - \beta + \alpha + \alpha + \beta = 12$$

$$3\alpha = 12$$

$$\alpha = 4$$

4	1	-12	39	-28
	0	4	-32	28
	1	-8	7	0

$$x^2 - 8x + 7 = 0$$

$$x^2 - 7x - x + 7 = 0$$

$$(x-1)(x-7) = 0$$

The roots are x = 4, 1 and 7.

3. *If* the roots of the *equation* $x^3 + px^2 + qx + r = 0$ *are in A.P*, then show that $2p^3 - 9pq + 27r = 0$

Solution

Let $\alpha - \beta, \alpha, \alpha + \beta$ be the roots

$$\alpha - \beta + \alpha + \alpha + \beta = -p$$

$$3\alpha = -p$$

$$\alpha = -\frac{p}{3}$$

α is one of the root

Substituting α in the given equation, we get

$$(-\frac{p}{3})^3 + p(-\frac{p}{3})^2 + q(-\frac{p}{3}) + r = 0$$

$$-\frac{p^3}{27} + p(-\frac{p^2}{9}) - (\frac{pq}{3}) + r = 0$$

$$-p^3 + 3p^3 - 9pq + 27r = 0$$

$$2p^3 - 9pq + 27r = 0$$

4. *Find the* value of k so that the roots of the *equation* $2x^3 + 6x^2 + 5x + k = 0$ *are in A.P*

Solution

Let $\alpha - \beta, \alpha, \alpha + \beta$ be the roots

$$\alpha - \beta + \alpha + \alpha + \beta = \frac{-6}{2}$$

$$3\alpha = -3$$

$$\alpha = -1$$

-1	2	6	5	k
	0	-2	-4	-1
	2	4	1	k=1

ANOTHER METHOD

$$2x^3 + 6x^2 + 5x + k = 0$$

$$2(-1)^3 + 6(-1)^2 + 5(-1) + k = 0$$

$$-2+6-5+k = 0$$

$$k = 1$$

5. *Solve the equation* $4x^3 - 36x^2 + 59x + 39 = 0$ *given that the roots are i n A P*

Solution

Let $\alpha - \beta, \alpha, \alpha + \beta$ be the roots

$$\alpha - \beta + \alpha + \alpha + \beta = \frac{36}{4}$$

$$3\alpha = 9$$

$$\alpha = 3$$

3	4	-36	59	39
	0	12	-72	-39
	4	-24	-13	0

$$4x^2 - 24x - 13 = 0$$

$$4x^2 + 2x - 26x - 13 = 0$$

$$2x(2x+1) - 13(2x+1) = 0$$

$$(2x-13)(2x+1) = 0$$

The roots are x = $3, \frac{13}{2}$ and $\frac{-1}{2}$

5. *Solve the equation* $2x^3 - 21x^2 + 42x - 16 = 0$ *given that the roots are i n*G. *P*

Solution

Let $\frac{a}{r}, a, ar$ be the roots

$$\frac{a}{r}.a.ar = -(\frac{-16}{2}) = 8$$

$$a^3 = 8$$

$$a = 2$$

2	2	-21	42	-16
	0	4	-34	16
	2	-17	8	0

$$2x^3 - 17x + 8 = 0$$

$$2x^2 - 16x - x + 8 = 0$$

$$2x(x-8) - 1(x-8) = 0$$

$$(2x-1)(x-8) = 0$$

The roots are x = 2,8 and $\frac{1}{2}$

6. *Solve the equation* $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$ *given that the roots are in* A.*P*

Solution

Let $\alpha, \beta, \gamma, \delta$ be the roots

The roots are in A.P

$$\alpha + \delta = \beta + \gamma$$

$$x^4 - 8x^3 + 14x^2 + 8x - 15 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

$$= [x^2 - (\alpha+\delta)x + \alpha\delta][x^2 - (\beta+\gamma)x + \beta\gamma]$$

$$= x^4 - (\beta+\gamma)x^3 + \beta\gamma x^2 - (\alpha+\delta)x^3 + (\alpha+\delta)(\beta+\gamma)x^2 - (\alpha+\delta)\beta\gamma x + \alpha\delta x^2 - (\beta+\delta)\alpha\delta x + \alpha\beta\gamma\delta$$

$$= x^4 - [(\beta+\gamma)(\alpha+\delta)]x^3 + [\beta\gamma + (\alpha+\delta)(\beta+\gamma) + \alpha\delta]x^2 - [(\alpha+\delta)\beta\gamma + (\beta+\delta)\alpha\delta]x + \alpha\beta\gamma\delta$$

Comparing the coefficient of x^3

$-[(\beta+\gamma)(\alpha+\delta)]$ = -8

$$2(\alpha+\delta)=8$$
$$(\alpha+\delta)=4$$

Comparing the coefficient of x^2

$\beta\gamma+(\alpha+\delta)\ (\beta+\gamma)+\alpha\delta=14$

$\beta\delta+\alpha\delta+(\alpha+\delta)^2=14$
$\beta\delta+\alpha\delta+16=14$
$\beta\delta+\alpha\delta=-2$

Comparing the coefficient of x

$-[(\alpha+\delta)\beta\gamma+(\beta+\delta)\alpha\delta]=8$

$-[4\beta\gamma+4\alpha\delta]=8$
$-4(\beta\gamma+\alpha\delta)=8$
$\beta\gamma+\alpha\delta=-2$

Equating constant coefficients

$\alpha\beta\gamma\delta=-15$
$\alpha\delta+\beta\gamma=-2$

This equations gives us the sum of roots and product of roots

Sum of the roots = -2

Product of roots = -15

On solving these equations, we get

$\alpha\delta=-5, \beta\gamma=3$
$\alpha+\delta=4, \alpha=5$
$\alpha\delta=-5, \delta=-1$
$\beta+\delta=4, \beta=3$
$\beta\gamma=3, \gamma=1$

Therefore the roots are 5, -1, 1, 3.

7. *Solve the equation* $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$ *given that the roots are in* A. *P*

Solution

Let $\alpha, \beta, \gamma, \delta$ be the roots

The roots are in A.P

$\alpha + \delta = \beta + \gamma$

$x^4 - 2x^3 - 21x^2 + 22x + 40 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= [x^2 - (\alpha+\delta)x + \alpha\delta][x^2 - (\beta+\gamma)x + \beta\gamma]$

$= x^4 - (\beta+\gamma)x^3 + \beta\gamma x^2 - (\alpha+\delta)x^3 + (\alpha+\delta)(\beta+\gamma)x^2 - (\alpha+\delta)\beta\gamma x + \alpha\delta x^2 - (\beta+\delta)\alpha\delta x + \alpha\beta\gamma\delta$

$= x^4 - [(\beta+\gamma)(\alpha+\delta)]x^3 + [\beta\gamma + (\alpha+\delta)(\beta+\gamma) + \alpha\delta]x^2 - [(\alpha+\delta)\beta\gamma + (\beta+\delta)\alpha\delta]x + \alpha\beta\gamma\delta$

Comparing the coefficient of x^3

$-[(\beta+\gamma)(\alpha+\delta)] = -2$
$2(\alpha+\delta) = 2$
$(\alpha+\delta) = 1$

Comparing the coefficient of x^2

$\beta\gamma + (\alpha+\delta)(\beta+\gamma) + \alpha\delta = -21$

$\beta\gamma + \alpha\delta + (\alpha+\delta)^2 = -21$
$\beta\gamma + \alpha\delta + 1 = -21$
$\beta\gamma + \alpha\delta = -22$

Comparing the coefficient of x

$-[(\alpha+\delta)\beta\gamma + (\beta+\delta)\alpha\delta] = 22 = 8$

$\beta\gamma + \alpha\delta = 22$
$-1(\beta\gamma + \alpha\delta) = 22$
$\beta\gamma + \alpha\delta = -22$

Equating constant coefficients

$\alpha\beta\gamma\delta = 40$
$\alpha\delta + \beta\gamma = -22$

Sum of the roots = -22

Product of roots = 40

On solving these equations, we get

$\alpha\delta = -20, \beta\gamma = -2$
$\alpha + \delta = 1, \alpha = 5$
$\alpha\delta = -20, \delta = -4$
$\delta = -4, \beta = 2$
$\beta + \gamma = 1, \gamma = -1$

Therefore the roots are 5, -4, 2, -1.

8. *Solve the equation* $x^3 - 3x^2 - 4x + 12 = 0$ *given that* sum of two *root* is zero.

Solution

Let α, β, γ be the roots

The sum of two root is zero

$\alpha + \beta + \gamma = 3$
$\alpha + (-\alpha) + \beta = 3$
$\beta = 3$

$\alpha\beta\gamma = -12$
$\alpha(-\alpha)\beta = -12$
$-\alpha^2\beta = -12$
$\alpha^2\beta = 12$
$\alpha^2(3) = 12$
$\alpha^2 = 4$
$\alpha = 2$

The roots are 2, -2, 3

9. *Solve the equation* $x^3 - 4x^2 - 3x + 18 = 0$ *given that* two of its *roots* are equa l.

Solution

Let α, α, β be the roots

The sum of two root is zero

$\alpha + \alpha + \beta = 4$
$2\alpha + \beta = 4$
$\beta = 4 - 2\alpha$

$\alpha^2 + \alpha\beta + \alpha\beta = -3$
$\alpha^2 + 2\alpha\beta = -3$
$\alpha^2 + 2\alpha(4 - 2\alpha) = -3$
$-3\alpha^2 + 8\alpha + 3 = 0$
$3\alpha^2 - 8\alpha - 3 = 0$
$3\alpha^2 - 9\alpha + \alpha - 3 = 0$
$3\alpha(\alpha - 3) + 1(\alpha - 3) = 0$
$(3\alpha + 1)(\alpha - 3) = 0$
$\alpha = \frac{-1}{3}$
$\alpha = 3$

If $\alpha = \frac{-1}{3}, \beta = 4 - 2(\frac{-1}{3}) = \frac{4+2}{3} = \frac{12+2}{3} = \frac{14}{3}$

If $\alpha = 3, \beta = 4 - 2(3) = 4 - 6 = -2$

$\beta = -2$

For satisfying,

$\alpha.\alpha.\beta = -18$
$\alpha^2\beta = -18$
$(-\frac{1}{3})^2(\frac{14}{3}) = 18$
$\frac{14}{27}$ is not satisfy

$(3)^2(-2) = 9(-2) = -18$
$\alpha = 3, \beta = -2$ satisfy the condition
so 3, 3, -2 are the roots.

10. *Solve the equation* $x^4 + x^3 - 16x^2 - 4x + 48 = 0$ *given that* product of two *roots* is 6.

Solution

Let $\alpha, \beta, \gamma, \delta$ be the roots

The roots are in A.P

$\alpha\beta = 6$

$\alpha.\beta.\gamma.\delta = 48$
$6\gamma.\delta = 48$
$\gamma.\delta = 8$

$x^4 + x^3 - 16x^2 - 4x + 48 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= [x^2 - (\alpha+\beta)x + \alpha\beta][x^2 - (\gamma+\delta)x + \gamma\delta]$
$= [x^2 - (\alpha+\beta)x + 6][x^2 - (\gamma+\delta)x + 8]$

Comparing the coefficient of x^3

$-(\gamma+\delta) - (\alpha+\beta) = 1$
$[(\gamma+\delta) + (\alpha+\beta)] = -1$
$\alpha+\beta+\gamma+\delta = -1$
$\alpha+\beta = p; \gamma+\delta = q$
$p+q = -1$

Comparing the coefficient of x^2

$8+6+(\alpha+\beta)(\gamma+\delta) = -16$
$(\alpha+\beta)(\gamma+\delta) = -16-14$
$pq = -30$

P+q = -1; pq = -30

P = 5, q = -6

$\alpha+\beta = 5$
$\alpha = 3, \beta = 2$
$q = -6; \gamma+\delta = -6,$
$\gamma\delta = 8$
$\delta = -2, \gamma = -4$

Therefore the roots are 3, 2, -4, -2.

11. *Solve the equation* $6x^3 - 11x^2 + 6x - 1 = 0$ *given that* the *roots* are in H.P.

Solution

Since reciprocal of A.P is H.P

$1x^3 - 6x^2 + 11x - 6 = 0$, *now the roots* are in A.P.

Let $\alpha-\beta, \alpha, \alpha+\beta$ be the roots

$\alpha - \beta + \alpha + \alpha + \beta = 6$

$3\alpha = 6$

$\alpha = 2$

2	1	-6	11	-6
	0	2	-8	6
	1	-4	3	0

$x^2 - 4x + 3 = 0$

$x^2 - 3x - x + 3 = 0$

$x(x-3) - 1(x-3) = 0$

$(x-1)(x-3) = 0$

The roots are x = 2, 1, 3 in A.P

The roots are in H.P are $1, \frac{1}{2}, \frac{1}{3}$

PROBLEMS FOR PRACTICE

1. *Solve the equation* $x^4 + 2x^3 - 25x^2 - 26x + 120 = 0$ *given that* product of two *roots* is 8.
clue:let $\alpha,\beta,\gamma,\delta$.and $\alpha\beta=8,\gamma\delta=15$
answer :$\alpha=2,\beta=4,\gamma=-3,\delta=-5$

2. *Solve the equation* $x^4 - 4x^3 - 17x^2 + 24x + 36 = 0$ *given that* product of two *roots* is 12.
clue: $\alpha, \beta, \gamma, \delta$. and $\alpha\beta = 12$, $\gamma\delta = 3$
answer :$\alpha=6,\beta=2,\gamma=-3,\delta=-1$
3. *Solve the equation* $x^4 - 4x^3 - 26x^2 + 8x + 5 = 0$ *given that* product of two *roots* is unity.
4. *Solve the equation* $x^3 - 9x^2 + 14x + 24 = 0$ *given that* two of its *roots* are in the ratio 3:2.
clue: $3\alpha, 2\alpha, \beta$ are the roots.
answer :$\alpha=2,\beta=-1$.
5. *Solve the equation* $2x^3 - x^2 - 22x - 24 = 0$ *given that* two of its *roots* are in the ratio 3:4.
6. *Solve the equation* $15x^4 - 8x^3 - 14x^2 + 8x - 1 = 0$ *given that* the *roots* are in H.P.

3.5 IMAGINARY AND IRRATIONAL ROOTS

In an equation with real coefficients, imaginary roots occur in pairs.

Let the equation be $f(x)=0$ and let $\alpha+i\beta$ be an imaginary root of the equation. We shall show that $\alpha-i\beta$ is also a root.

We have $(x-\alpha-i\beta)(x-\alpha+i\beta)=(x-\alpha)^2+\beta^2---(1)$

If $f(x)$ is divided by $(x-\alpha)^2+\beta^2$, let the quotient be $Q(x)$ and the remainder be $Rx+R'$. Here $Q(x)$ is of degree $(n-2)$

Therefore $f(x)=\{(x-\alpha)^2+\beta^2\}Q(x)+Rx+R'-----(2)$

Substituting $\alpha+i\beta$ for x in the equation (2), we get

$\mathbf{f}(\alpha+i\beta)=\{((\alpha+i\beta)-\alpha)^2+\beta^2\}Q(\alpha+i\beta)+R(\alpha+i\beta)+R'$

$= R(\alpha+i\beta)+R'$

But $f(\alpha+i\beta)=0$ since $\alpha+i\beta$ is a root of $f(x)=0$

Therefore $R(\alpha+i\beta)+R'=0$

Equating to zero the real and imaginary parts $R\alpha+R'=0$ *and* $R\beta = 0$

Since $\beta\neq 0$, R=0 and so $R' = 0$

$f(x)=\{(x-\alpha)^2+\beta^2\}Q(x)$

Therefore $\alpha-i\beta$ is also a root of $f(x)=0$.

Example1

Form a rational cubic equation which shall have for roots $1, 3-\sqrt{-2}$

Solution

Since $3-\sqrt{-2}$ is a root of the equation, $3+\sqrt{-2}$ is also a root.

So we have to form an equation whose roots are $1, 3-\sqrt{-2}, 3+\sqrt{-2}$.

Hence the required equation is

$(x-1)(x-3+\sqrt{-2})(x-3-\sqrt{-2})=0$

$(x-1)\{(x-3)^2+2\}=0$

$(x-1)(x^2-6x+11)=0$

$x^3-7x^2+17x-11=0$

Example 2

Solve the equation $6x^4-13x^3-35x^2-x+3=0$ *given that* $2-\sqrt{3}$ *is a root.*

Solution

The given root is $2-\sqrt{3}$, then the other root is $2+\sqrt{3}$

Form the equation using the known roots

$(x-(2 - \sqrt{3}))(x - (2 + \sqrt{3})) = 0$

$((x-2)+ \sqrt{3})((x-2)- \sqrt{3}) = 0$

$((x-2)^2 - (\sqrt{3})^2)=0$

$x^2 -4x +4-3 = 0$

$x^2 - 4x+1 = 0-----(1)$

To find the other roots divide the given equation by (1)

Quotient: $6x^2 + 11x + 3$

Divisor: $x^2 - 4x + 1$

$6x^4 - 13x^3 - 35x^2 - x + 3$
$6x^4 - 24x^3 + 6x^2$

$11x^3 - 41x^2 - x$
$-11x^3 - 44x^2 + 11x$

$3x^2 - 12x + 3$
$3x^2 - 12x + 3$

0

On solving the equation

$6x^2 +11x +3= 0$

$6x^2 +2x + 9x +3= 0$

$2x(3x + 1)+3(3x + 1)= 0$

$(2x + 3)(3x+1)=0$

$x = -\frac{3}{2}, -\frac{1}{2}$ Therefore the roots are $x = -\frac{3}{2}, -\frac{1}{2}$, $2 - \sqrt{3}$, $2 + \sqrt{3}$

Example 3

Solve the equation $x^4 - 4x^3 + 8x + 35 = 0$ *given that* $2 + i\sqrt{3}$ *is a root.*

Solution

The given root is $2 - i\sqrt{3}$, then the other root is $2 + i\sqrt{3}$

Form the equation using the known roots

$(x-(2 - i\sqrt{3}))(x - (2 + i\sqrt{3})) = 0$

$$((x-2) - i\sqrt{3}))((x - 2) + i\sqrt{3})) = 0$$

$$((x-2)^2 - (i\sqrt{3})^2)=0$$

$$x^2 - 4x + 4 + 3 = 0$$

$$x^2 - 4x + 7 = 0-----(1)$$

To find the other roots divide the given equation by

$$\begin{array}{r|l} & x^2 + 4x + 5 \\ \hline x^2-4x+7 & x^4 - 0 - 4x^2 + 8x + 35 \\ & x^4 - 4x^3 + 7x^2 \\ \hline & 4x^3 - 11x^2 - 8x \\ & 4x^3 - 14x^2 + 28x \\ \hline & 5x^2 - 20x + 35 \\ & 5x^2 - 20x + 35 \\ \hline & 0 \end{array}$$

(1)

On solving the equation $x^2 + 4x + 5 = 0$

$x = 2 \pm i\sqrt{3}, 2 \pm i$

Therefore the roots are $x = 2 \pm i\sqrt{3}, 2 \pm i$.

Example 4

Solve the equation $x^4 - 11x^2 + 2x + 12 = 0$ *given that* $\sqrt{5} - 1$ *is a root.*

Solution

The given root is $\sqrt{5} - 1$, then the other root is $-\sqrt{5} - 1$

Form the equation using the known roots

$$(x - (\sqrt{5} - 1))(x - (\sqrt{5} - 1)) = 0$$

$$((x+1)-\sqrt{5})((x+1) + \sqrt{5}) = 0$$

$$((x+1)^2 - (\sqrt{5})^2) = 0$$

$$x^2 + \mathbf{2}x + \mathbf{1} + \mathbf{5} = 0$$

$$x^2 + \mathbf{2}x + 6 = 0----(1)$$

To find the other roots divide the given equation by

$x^2 - 2x - 3$

$x^2 + 2x - 4 \,\big|\, x^4 - 0 - 11x^2 + 2x + 12$

$x^4 + 2x^3 - 4x^2$

$-2x^3 - 7x^2 + 2x$

$-2x^3 - 4x^2 + 8x$

$-3x^2 - 6x + 12$

$-3x^2 - 6x + 12$

0

(1)

On solving the equation

$$x^2 - 2x - 3 = 0$$

$$x^2 - 3x + x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$x = -1,\ 3$$

Therefore the roots are $x = -1 \pm \sqrt{5}\ , -1, 3.$

Example 5

Show that $\frac{a^2}{x-\alpha} + \frac{b^2}{x-\beta} + \frac{c^2}{x-\gamma} - x + \delta = 0$ *has only real roots if* $a,\ b,\ c,\ \alpha,\ \beta,\ \gamma,\ \delta$ *are all real.*

Solution

If possible let $p + iq$ be a root. Then $p - iq$ is also a root.

$$\frac{a^2}{p+iq-\alpha} + \frac{b^2}{p+iq-\beta} + \frac{c^2}{p+iq-\gamma} - (p+iq) + \delta = 0 ----(1)$$

$$\frac{a^2}{p-iq-\alpha} + \frac{b^2}{p-iq-\beta} + \frac{c^2}{p-iq-\gamma} - (p-iq) + \delta = 0 ----(2)$$

Subtracting (2) from (1), we get

$$a^2\left[\frac{1}{(p-\alpha)+iq} - \frac{1}{(p-\alpha)-iq}\right] + b^2\left[\frac{1}{(p-\beta)+iq} - \frac{1}{(p-\beta)-iq}\right]$$

$$+c^2\left[\frac{1}{(p-\gamma)+iq} - \frac{1}{(p-\gamma)-iq}\right] - 2iq = 0$$

$$-2iq\left[\frac{a^2}{(p-\alpha)^2+q^2} + \frac{b^2}{(p-\beta)^2+q^2} - \frac{c^2}{(p-\gamma)^2+q^2} + 1\right] = 0$$

Since the expression within the square bracket is not equal to zero, 2iq = 0.

Therefore q = 0. Hence all the roots of the equation are real.

PROBLEMS FOR PRACTICE

1. *Solve the equation* $2x^3 - 7x^2 + 4x + 3 = 0$ *given that* $1 + \sqrt{2}$ *is a root.*

answer: *roots* $1 \pm \sqrt{2}$ *and* $\frac{3}{2}$.

2. *Solve the equation* $x^5 - x^4 + 8x^2 - 9x - 15 = 0$ *given that* $1 + 2\sqrt{-1}$ *and* $(-\sqrt{3})$ *are the roots. clue* : $1 + 2\sqrt{-1}$ *can be written as* $1 + 2i$ *Answer*: $1 \pm 2i, \pm\sqrt{3}, -1$

3. *Form the equation whose roots are* $\pm\sqrt{3}, \sqrt{2} \pm i.$

clue: *roots are* $\sqrt{3}, -\sqrt{3}, \sqrt{2}+i, \sqrt{2}-i, -\sqrt{2}+i, -\sqrt{2}-i.$

Answer : $x^6 - 5x^4 + 15x^2 - 27 = 0$

4. *Form the equation, given that* $\sqrt{2} + \sqrt{3}$ *is a root.*

clue: *roots are* $-\sqrt{2} + \sqrt{3}, -\sqrt{2} - \sqrt{3}, \sqrt{2} + \sqrt{3}, \sqrt{2} - \sqrt{3}$

Answer : $x^4 - 10x^2 + 1 = 0$

3.6 SYMMETRIC FUNCTION OF THE ROOTS

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged, it is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ be the roots of the equation.

$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + ... + p_n = 0$

We have learned that

$S_1 = \sum \alpha_1 = -p_1$

$S_2 = \sum \alpha_1 \alpha_2 = p_2$

$S_3 = \sum \alpha_1 \alpha_2 \alpha_3 = -p_3$

....

....

Without knowing the values of the roots separately in terms of the coefficients, by using the above relations between the coefficients and the roots of an equation, we can express any symmetric function of the roots in terms of the coefficients of the equations.

Example 1

If are the roots of the equation $x^3 + px^2 + qx + r = 0$, exp*ress the value of* $\sum \alpha^2 \beta$ *in terms of the coefficients.*

Solution

We have $\alpha+\beta+\gamma=-p$

$$\alpha\beta+\beta\gamma+\gamma\alpha=q$$

$$\alpha\beta\gamma=-r$$

$$\begin{aligned}\sum\alpha^2\beta &= \alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\\ &= \left(\alpha\beta+\beta\gamma+\gamma\alpha\right)(\alpha+\beta+\gamma)-3\alpha\beta\gamma\\ &= q(-p)-3(-r)\\ &= 3r-pq\end{aligned}$$

Example 2

If $\alpha,\beta,\gamma,\delta$ *be the roots of the biquadratic equation* $x^4+px^3+qx^2+rx+s=0,$ *find* $\sum\alpha^2$

Solution

We have $\alpha+\beta+\gamma+\delta=-p$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-r$$

$$\alpha\beta\gamma\delta=s$$

$$\begin{aligned}\sum\alpha^2 &= \alpha^2+\beta^2+\gamma^2+\delta^2\\ &= (\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)\\ &= (\sum\alpha)^2-2(\sum\alpha\beta)\\ &= p^2-2q.\end{aligned}$$

Example 3

If $\alpha,\beta,\gamma,\delta$ *be the roots of the biquadratic equation* $x^4+px^3+qx^2+rx+s=0,$ *find* $\sum\alpha^2\beta\gamma$

Solution

We have $\alpha+\beta+\gamma+\delta=-p$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-r$$

$$\alpha\beta\gamma\delta=s$$

$$\begin{aligned}\sum\alpha^2\beta\gamma &= (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)(\alpha+\beta+\gamma+\delta)-4\alpha\beta\gamma\delta\\ &= \left(\sum\alpha\beta\gamma\right)\left(\sum\alpha\right)-4\alpha\beta\gamma\delta\\ &= pr-4s.\end{aligned}$$

Example 4

If $\alpha, \beta, \gamma, \delta$ *be the roots of the biquadratic equation* $x^4 + px^3 + qx^2 + rx + s = 0,$ *find* $\sum \alpha^2\beta^2$

Solution

We have $\alpha + \beta + \gamma + \delta = -p$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$$
$$\alpha\beta\gamma\delta = s$$

$$\begin{aligned}\sum \alpha^2\beta^2 &= \alpha^2\beta^2 + \alpha^2\delta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2 \\ &= \left(\sum \alpha\beta\right)^2 - 2\left(\sum \alpha^2\beta\gamma\right) - 6\alpha\beta\gamma\delta \\ &= q^2 - 2(pr - 4s) - 6s \\ &= q^2 - 2pr + 2s.\end{aligned}$$

Example 5

If $\alpha, \beta, \gamma, \delta$ *be the roots of the biquadratic equation* $x^4 + px^3 + qx^2 + rx + s = 0,$ *find* $\sum \alpha^3\beta$

Solution

We have $\alpha + \beta + \gamma + \delta = -p$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$$
$$\alpha\beta\gamma\delta = s$$

$$\begin{aligned}\sum \alpha^3\beta &= \left(\sum \alpha^2\right)\left(\sum \alpha\beta\right) - \left(\sum \alpha^2\beta\gamma\right) \\ &= (\mathrm{p}^2 - 2q)q - (pr - 4s) \\ &= \mathrm{p}^2 q - 2q^2 - pr + 4s.\end{aligned}$$

Example 6

If $\alpha, \beta, \gamma, \delta$ *be the roots of the biquadratic equation* $x^4 + px^3 + qx^2 + rx + s = 0,$ *find* $\sum \alpha^4,$

Solution

We have $\alpha + \beta + \gamma + \delta = -p$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q$$
$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-r$$
$$\alpha\beta\gamma\delta=s$$

$$\sum\alpha^4=\left(\sum\alpha^2\right)^2-2\left(\sum\alpha^2\beta^2\right)$$
$$=(p^2-2q)^2-2(q^2-2pr+2s)$$
$$=p^4-4p^2q+2q^2+4pr-4s.$$

Example 7

If α,β,γ are the roots of the equation $x^3+ax^2+bx+c=0$, then form the equation whose roots are $\alpha\beta,\beta\gamma$ and $\gamma\alpha$.

Solution

The relations between the roots and coefficients are

$$\alpha+\beta+\gamma=-a$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=b$$
$$\alpha\beta\gamma=-c$$

The required equation is

$$(x-\alpha\beta)(x-\beta\gamma)(x-\gamma\alpha)=0$$
$$x^3-x^2(\alpha\beta+\beta\gamma+\gamma\alpha)+(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)x-\alpha^2\beta^2\gamma^2=0$$
$$x^3-x^2(\alpha\beta+\beta\gamma+\gamma\alpha)+x\alpha\beta\gamma(\alpha+\beta+\gamma)-(\alpha\beta\gamma)^2=0$$
$$x^3-bx^2+acx-c^2=0$$

Example 8

If α,β,γ are the roots of the equation $x^3+px^2+qx+r=0$, then form the equation whose roots are $\beta+\gamma-2\alpha,\gamma+\alpha-2\beta$ and $\alpha+\beta$-2γ.

Solution

The relations between the roots and coefficients are

$$\alpha+\beta+\gamma=-p$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=q$$
$$\alpha\beta\gamma=-r$$

In the required equation

$S_1=$ Sum of the roots $=\beta+\gamma-2\alpha+\gamma+\alpha-2\beta+\alpha+\beta\text{-}2\gamma=0.$

$S_2=$ Sum of the products of the roots taken two at a time

$=(\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)+(\alpha+\beta\text{-}2\gamma)(\beta+\gamma-2\alpha)+(\gamma+\alpha-2\beta)(\alpha+\beta\text{-}2\gamma)$

$=(\alpha+\beta+\gamma-3\alpha)(\alpha+\beta+\gamma-3\beta)+2$ similar terms

$= (-p-3\alpha)(-p-3\beta)+(-p-3\alpha)(-p-3\gamma)+(-p-3\gamma)(-p-3\beta)$

$= (p+3\alpha)(p+3\beta)+(p+3\alpha)(p+3\gamma)+(p+3\gamma)(p+3\beta)$

$= 3p^2+6p(\alpha+\beta+\gamma)+9(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= 3p^2+6p(-p)+9(q)$

$= 9q-3p^2$

$S_3 =$ Products of the roots

$= (\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)(\alpha+\beta-2\gamma)$

$= (\alpha+\beta+\gamma-3\alpha)(\alpha+\beta+\gamma-3\beta)(\alpha+\beta+\gamma-3\gamma)$

$= (-p-3\alpha)(-p-3\beta)(-p-3\gamma)$

$= -\{p^3+3p^2(\alpha+\beta+\gamma)+9p(\alpha\beta+\beta\gamma+\gamma\alpha)+27\alpha\beta\gamma\}$

$= -\{p^3+3p^2(-p)+9pq-27r\}$

$= 2p^3-9pq+27r$

Hence the required equation is

$x^3-S_1x^2+S_2x-S_3=0 \quad x^3+(9q-3p^2)x-(2p^3-9pq+27r)=0$

PROBLEMS FOR PRACTICE

1. *If* α,β,γ *be the roots of the equation* $x^3+px^2+qx+r=0$, *find the value of* $\sum\frac{1}{\alpha^2}$.

answer: $\frac{q^2-2pr}{r^2}$

2. *If* α,β,γ *be the roots of the equation* $x^3+px^2+qx+r=0$, *find the value of* $\sum\alpha^3$.

answer: $3pq-p^3-3r$

3. *If* α,β,γ *be the roots of the equation* $x^3+px^2+qx+r=0$, *prove that* $\sum(\alpha+\beta)=r-pq$.

4. *If* α,β,γ *be the roots of the equation* $x^3-3ax+b=0$, *prove that* $\sum(\alpha-\beta)(\alpha-\gamma)=9a$.

5. *If* $\alpha,\beta,\gamma,\delta$ *be the roots of the biquadratic equation* $x^4+px^3+qx^2+rx+s=0$,

find (i) $\sum\alpha^2\beta$ (ii) $\sum(\beta+\gamma+\delta)^2$ (iii) $\sum\frac{1}{\alpha^2}$

answer: (i) $3r-pq$ (ii) $3p^2-2q$ (iii) $\frac{r^2-2qs}{s^2}$

$6. If\ \alpha,\beta,\gamma$ be the roots of the equation $x^3+px^2+qx+r=0$, find the value of

$$(i)\frac{\beta^2+\gamma^2}{\beta+\gamma}+\frac{\gamma^2+\alpha^2}{\gamma+\alpha}+\frac{\alpha^2+\beta^2}{\alpha+\beta}$$

$$(i)\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$answer:(i)\frac{2p^2q-4pr-2q^2}{r-pq}(ii)\frac{pq-3r}{r}$$

3.7 SUM OF POWERS OF THE ROOTS OF AN EQUATION

Let $\alpha_1,\alpha_2,\alpha_3,......,\alpha_n$ be the roots of the equation $f(x)=0$.

We have $f(x)=(x-\alpha_1)(x-\alpha_2)......(x-\alpha_n)$

Taking logarithm on both sides and differentiating, we get

$$\frac{f'(x)}{f(x)}=\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+......+\frac{1}{x-\alpha_n}$$

$$\frac{xf'(x)}{f(x)}=\frac{1}{1-\frac{\alpha_1}{x}}+\frac{1}{1-\frac{\alpha_2}{x}}+......+\frac{1}{1-\frac{\alpha_n}{x}}$$

$$\frac{xf'(x)}{f(x)}=\left(1-\frac{\alpha_1}{x}\right)^{-1}+\left(1-\frac{\alpha_2}{x}\right)^{-1}+......+\left(1-\frac{\alpha_n}{x}\right)^{-1}$$

$$\frac{xf'(x)}{f(x)}=\left(1+\frac{\alpha_1}{x}+\left(\frac{\alpha_1}{x}\right)^2+....+\left(\frac{\alpha_1}{x}\right)^n+.....\right)$$

$$+\left(1+\frac{\alpha_2}{x}+\left(\frac{\alpha_2}{x}\right)^2+....+\left(\frac{\alpha_2}{x}\right)^n+.....\right)+.....$$

....

$$+\left(1+\frac{\alpha_n}{x}+\left(\frac{\alpha_n}{x}\right)^2+....+\left(\frac{\alpha_n}{x}\right)^n+.....\right)+.....$$

$$=n+S_1.\frac{1}{x}+S_2.\frac{1}{x^2}+.......+S_r.\frac{1}{x^r}+.....$$

S_r = Coefficient of $\frac{1}{x^r}$ in the expansion of $\frac{xf'(x)}{f(x)}$

Example

Find the sum of the cubes of the roots of the equation $x^5 = x^2 + x + 1$

Solution

The equation can be written in the form $f(x) = x^5 - x^2 - x - 1 = 0$

$$S_3 = Cofficient\ of\ \frac{1}{x^3}\ in\ the\ \exp ansion\ of\ \frac{x(5x^4 - 2x - 1)}{x^5 - x^2 - x - 1}$$

$$= Cofficient\ of\ \frac{1}{x^3}\ in\ \frac{(5 - \frac{2}{x^3} - \frac{1}{x^4})}{1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5}}$$

$$= Cofficient\ of\ \frac{1}{x^3}\ in\ (5 - \frac{2}{x^3} - \frac{1}{x^4})\left(1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5}\right)^{-1}$$

$$= Cofficient\ of\ \frac{1}{x^3}\ in\ (5 - \frac{2}{x^3} - \frac{1}{x^4})\left[\left(1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5}\right) + \left(1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5}\right)^2 + ...\right]$$

$$= Cofficient\ of\ \frac{1}{x^3}\ in\ (5 - \frac{2}{x^3} - \frac{1}{x^4})(1 - \frac{1}{x^3} + ...)$$

$$= 3.$$

3.8 NEWTON'S THEOREM ON THE SUM OF THE POWERS OF THE ROOTS

Let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ be the roots of the equation $f(x) = 0$.

We have $f(x) = (x - \alpha_1)(x - \alpha_2)......(x - \alpha_n)$

Taking logarithm on both sides and differentiating, we get

$$\frac{f'(x)}{f(x)} = \frac{1}{x - \alpha_1} + \frac{1}{x - \alpha_2} + ... + \frac{1}{x - \alpha_n}$$

$$f'(x) = \frac{f(x)}{x - \alpha_1} + \frac{f(x)}{x - \alpha_2} + ... + \frac{f(x)}{x - \alpha_n}$$

By Actual division, we obtain

$$\frac{f(x)}{x-\alpha_1} = x^{n-1} + (\alpha_1 + p_1)x^{n-2} + (\alpha_1{}^2 + p_1\alpha_1 + p_2)x^{n-3} + \dots\dots + (\alpha_1^{n-1} + p_1\alpha_1^{n-2} + \dots\dots + p_{n-1})$$

$$\frac{f(x)}{x-\alpha_2} = x^{n-1} + (\alpha_2 + p_1)x^{n-2} + (\alpha_1{}^2 + p_1\alpha_2 + p_2)x^{n-3} + \dots\dots + (\alpha_2^{n-1} + p_1\alpha_2^{n-2} + \dots\dots + p_{n-1})$$

....

$$\frac{f(x)}{x-\alpha_n} = x^{n-1} + (\alpha_n + p_1)x^{n-2} + (\alpha_n{}^2 + p_1\alpha_n + p_2)x^{n-3} + \dots\dots + (\alpha_n^{n-1} + p_1\alpha_n^{n-2} + \dots\dots + p_{n-1})$$

Adding all these fractions, we get $f'(x) = nx^{n-1} + (s_1 + np_1)x^{n-2} + (s_2 + p_1s_1 + np_2)x^{n-3} + \dots\dots + (s_{n-1} + p_1s_{n-2} + \dots\dots + np_{n-1})$

But $f'(x)$ is also equal to

$$nx^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \dots\dots + 2p_{n-2} + p_{n-1}$$

Equating the coefficient in the two values of $f'(x)$, we obtain the following relations:

$$s_1 + p_1 = 0$$
$$s_2 + p_1s_1 + 2p_2 = 0$$
$$s_3 + p_1s_2 + p_2s_1 + 3p_3 = 0$$
$$s_4 + p_1s_3 + p_2s_2 + p_3s_1 + 4p_4 = 0$$
....
....
$$s_{n-1} + p_1s_{n-2} + p_2s_{n-3} + \dots\dots + p_{n-2}s_1 + (n-1)p_{n-1} = 0$$

From these (n-1) relations we can calculate in succession the values of $s_1, s_2, \dots, s_{n-1}$ in terms of coefficients $p_1, p_2, \dots\dots, p_{n-1}$. We can extend our results to the sums of all positive powers of the roots, $s_n, s_{n+1}, \dots\dots, s_r$ where $r \succ n$.

We have $x^{r-n}f(x) = x^r + p_1x^{r-1} + p_2x^{r-2} + \dots\dots\dots + p_nx^{r-n}$

Replacing in this identity, x by the roots $\alpha_1, \alpha_2, \dots\dots, \alpha_n$, in succession and adding, we have $s_r + p_1s_{r-1} + p_2s_{r-2} + \dots\dots + p_ns_{r-n} = 0$

Now giving r the values $n, n+1, n+2, \dots\dots$ successively and observing that $S_0 = n$, we obtain from the last equation

$$s_n + p_1s_{n-1} + p_2s_{n-2} + \dots\dots + np_n = 0$$
$$s_{n+1} + p_1s_n + p_2s_{n-1} + \dots\dots + p_ns_1 = 0$$
....

Thus we get $\begin{matrix} s_r + p_1 s_{r-1} + p_2 s_{r-2} + ...+ r p_r = 0 \ if \ r \prec n \\ s_r + p_1 s_{r-1} + p_2 s_{r-2} + ...+ p_n s_{r-n} = 0 \ if \ r \geq n \end{matrix}$

Example 1

Show that the sum of the eleventh powers of the positive of the roots of $x^7 + 5x^4 + 1 = 0$ *is zero.*

Solution

Since $11 \succ 7$, the degree of the equation, we have to use the latter equation in Newton's theorem.If we assume the equation as $x^7 + p_1 x^6 + p_2 x^5 + p_3 x^4 + p_4 x^3 + p_5 x^2 + p_6 x + p_7 = 0$
We have $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 0, p_3 = 5, p_7 = 1$
$s_{11} + p_1 s_{10} + p_2 s_9 + p_3 s_8 + p_4 s_7 + p_5 s_6 + p_6 s_5 + p_7 s_4 = 0$
$s_{11} + 5s_8 + s_4 = 0 ------(1)$
Again $\begin{matrix} s_8 + p_1 s_7 + p_2 s_6 + p_3 s_5 + p_4 s_4 + p_5 s_3 + p_6 s_2 + p_7 s_1 = 0 \\ s_8 + 5s_5 + s_1 = 0 ------(2) \end{matrix}$

Using the first equation in the Newton's theorem

$s_5 + p_1 s_4 + p_2 s_3 + p_3 s_2 + p_4 s_1 + 5p_5 = 0$
$s_5 + 5s_2 = 0 ------(3)$

$s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1 + 4p_4 = 0$
$s_4 + 5s_1 = 0 ------(4)$

$s_2 + p_1 s_1 + 2p_2 = 0$
$s_2 = 0 ------(5)$

$s_1 = 0 ------(6)$

From (4), (5) and (6), we get $s_4 = 0$
From (4), (5) and (6), we get $s_5 = 0$
From (4), (5) and (6), we get $s_5 = 0$
From (4), (5) and (6), we get $s_8 = 0$
Substituting the values of s_4, s_8 in (1), we get $s_{11} = 0$.

Example 2

Find $\frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5}$ *where* α, β, γ *are the roots of the equation* $x^3 + 2x^2 - 3x - 1 = 0$

Solution

Put $x=\frac{1}{y}$ in the equation, then the equation becomes

$$\frac{1}{y^3}+\frac{2}{y^2}-\frac{3}{y}-1=0$$

$$y^3+3y^2-2y-1=0$$

The roots of the equation are $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$

Therefore $\frac{1}{\alpha^5}+\frac{1}{\beta^5}+\frac{1}{\gamma^5}=s_5$ for the equation $y^3+3y^2-2y-1=0$

From Newton's theorem on the sum of the powers of the roots of the equations, we get

$$s_5+3s_4-2s_3-s_2=0$$

$$s_4+3s_3-2s_2-s_1=0$$

$$s_3+3s_2-2s_1-s_0=0$$

$$s_2+3s_1-4=0$$

$$s_1+3=0$$

$$s_1=-3, s_2=13, s_3=-42, s_4=149, s_5=-518.$$

$$\frac{1}{\alpha^5}+\frac{1}{\beta^5}+\frac{1}{\gamma^5}=-518$$

PROBLEMS FOR PRACTICE

1.*If* $a+b+c+d=0$, *show that* $\frac{a^5+b^5+c^5+d^5}{5}=\frac{a^2+b^2+c^2+d^2}{2}.\frac{a^3+b^3+c^3+d^3}{3}$

2.*If* α,β,γ *are the roots of the equation* $x^3+px^2+qx+r=0$, *calculat e the value of* $\alpha^3+\beta^3+\gamma^3$

3.*Show that the sum of the fourth powers of the equa tion* $x^5+px^3+qx^2+s=0$ *is* $2p^2$.

4. *If* α,β,γ *are the roots of the equation* $x^3+qx+r=0$ *show that*

i) $3s_2s_5=5s_3s_4$

ii) $\frac{\alpha^5+\beta^5+\gamma^5}{5}=\frac{\alpha^2+\beta^2+\gamma^2}{2}.\frac{\alpha^3+\beta^3+\gamma^3}{3}$

iii) $\frac{\alpha^7+\beta^7+\gamma^7}{7}=\frac{\alpha^2+\beta^2+\gamma^2}{2}.\frac{\alpha^5+\beta^5+\gamma^5}{5}$

3.9 MULTIPLE ROOTS

If $f(x)=0$ has a multiple root α of multiplicity r, then $f'(x)=0$ has a multiple roots α of multiplicity $(r-1)$.

Let the degree of $f(x)=0$ be n.

Then $f(x)=(x-\alpha)^r\phi(x)$ where $\phi(\alpha)\neq 0$

Therefore $f'(x)=(x-\alpha)^r\phi'(x)+r(x-\alpha)^{r-1}\phi(x)$

$$=(x-\alpha)^{r-1}[r\phi(x)+(x-\alpha)\phi'(x)]=(x-\alpha)^{r-1}\chi(x)$$

Evidently, $\chi(\alpha)=r\phi(\alpha)\neq 0$

Therefore $f'(x)=0$ has a root of multiplicity $(r-1)$.

Example

1.*Solve* $x^3-4x^2+5x-2=0$ *given that it has a double root.*

Solution

Let $f(x)=x^3-4x^2+5x-2=0$ possess a double root α.

Then $f'(x)=3x^2-8x+5=0$ has a root α.

$3x^2-8x+5=(x-1)(3x-5)=0$

$x=1$ or $\frac{5}{3}$

$f(1)=1-4+5-2=0$

$x=1$ is the double root

$f(x)=(x-1)^2(x-2)$

The roots are 1, 1, 2.

2.*Solve* $8x^4+4x^3-18x^2+11x-2=0$ *given that it has three equal roots.*

Solution

Let $f(x)=8x^4+4x^3-18x^2+11x-2=0$ has roots $\alpha,\alpha,\alpha,\beta$.

Then $f'(x)=32x^3+12x^2-36x+11=0$ has a root α, α, γ.

$f''(x)=96x^2+24x-36=0$ has roots α, δ

Solve $96x^2+24x-36=0$ or $8x^2+2x-3=0$

$(4x+3)(2x-1)=0$

$x=\frac{1}{2}$ or $-\frac{3}{4}$

$f(\frac{1}{2})=8\left(\frac{1}{16}\right)+4\left(\frac{1}{8}\right)-18\left(\frac{1}{4}\right)+\frac{11}{2}-2$

$=\frac{1}{2}+\frac{1}{2}-\frac{9}{2}+\frac{11}{2}-2=0$

Therefore $\frac{1}{2}$ is the repeated root.

Product of roots = $\alpha^3\beta=\frac{1}{8}\beta=\frac{-2}{8}$

$\beta=-2$

Roots are $\frac{1}{2},\frac{1}{2},\frac{1}{2}$ and -2.

3. *Solve* $x^4+4x^3-2x^2-12x+9=0$ *given that it has two pairs of equal roots*

Solution

Let the equation posses the roots $\alpha, \alpha, \beta, \beta$

Sum of roots $2(\alpha+\beta)=-4$

$\therefore \alpha+\beta=-2$.

Then $f'(x)=4x^3+12x^2-4x-12=0$ has roots α, β, γ.

Sum of roots $\therefore \alpha+\beta+\gamma=-\frac{12}{4}=-3$

$\therefore -2+\gamma=-3$

$\therefore \gamma=-1$

Now solve $f'(x)=0$ knowing one root $\gamma=-1$

-1	4	12	-4	-12
		-4	-8	12
	4	8	-12	0

$4x^2+8x-12=0 (or) (x+3)(x-1)=0$

$x=1$ or -3

The roots are 1, 1, -3,-3.

4. *Find* k *so that the equation* $2x^3-9x^2+12x+k=0$ *may have a double root*

Solution

Let $f(x)=2x^3-9x^2+12x+k$

If $f(x) = 0$ has a double root, then $f'(x) = 0$ has that root.

Then $f'(x)=0$ *implies* $6x^2-18x+12=0$

$x^2-3x+2=(x-1)(x-2)=0$

$x=1$ or 2

If $x=1$ is the double root then f(1) = 0

$2-9+12+k=0$ $\therefore$ k = -5

If $x=2$ is the double root then f(2) = 0

$16-36+24+k=0$ $\therefore$ k = -4

PROBLEMS FOR PRACTICE

1. *Solve* $4x^3+8x^2+5x+1=0$ *given that it has a double root.*

$answer: x=-\frac{1}{2},-\frac{1}{2},-1$

1. *Solve* $8x^4+4x^3-18x^2+11x-2=0$ *given that it has three equal roots.*

Hint: Solve $f''(x)=0$ *and proceed. answer:* $\frac{1}{2},\frac{1}{2},\frac{1}{2},-2$

3. *Solve* $x^4-6x^3-13x^2-12x+4=0$ *given that it has two sets of equal roots*

Answer: 1, 1, 2, 2

4.*Solve* $x^4 - 14x^3 + 73x^2 - 168x + 144 = 0$ *given that it has a two pairs of equal roots.*

answer : 3, 3, 4, 4

5.*Solve* $2x^3 - 9x^2 + 12x - 4 = 0$ *given that it has two equal roots.*

answer : $2, 2, \frac{1}{2}$

6.*Solve* $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$ *given that it has two pairs of equal roots.*

answer : 1, 1, 3, −3

7.*If the equation* $x^3 - 3px^2 + 3qx - r = 0$ *has a pair of equal roots,*

show that the third root is $\dfrac{3p^3 - 4pq + r}{p^2 - q}$

Chapter 4

Transformation of Equations

Contents

4.0 AIMS AND OBJECTIVES

Our Aim is to learn about the Transformation of Equations. Further we aim at learning reciprocal equations, reciprocal roots and rule of sign for finding the roots.

4.1 TRANSFORMATION OF EQUATIONS

If an equation is given, it is possible to transform this equation into another whose roots bear with the roots of the original equation a given relation. Such a transformation often helps us to solve equations easily or to discuss the nature of the roots of the equations. We shall explain here the most important elementary transformations of equations.

4.2 RECIPROCAL ROOTS

If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are the the roots of the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + ...+ p_n = 0$ then the transform of this equation $p_n y^n + p_{n-1} y^{n-1} + ...+ p_2 y^2 + p_1 y + 1 = 0$ (by replacing $x = \frac{1}{y}$ and multiplying by y^n) has roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3} ......\frac{1}{\alpha_n}$.

4.3 RECIPROCAL EQUATIONS

If an equation $f(x) = 0$ unchanged while replacing $x = \frac{1}{y}$, then it is called a reciprocal equation.

4.3.1 CONDITION FOR AN EQUATION TO BE A RECIPROCAL EQUATION

A reciprocal equation of the standard form can always be depressed to another of half the dimension.

Let $f(x) = p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + ...+ p_{n-1} x + p_n = 0 --- (1)$ be a reciprocal equation. Then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3} ......\frac{1}{\alpha_n}$ are also the roots of (1). But $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3} ......\frac{1}{\alpha_n}$ are the roots of

$p_n y^n + p_{n-1} y^{n-1} + ...+ p_2 y^2 + p_1 y + 1 = 0 ----- (2)$

Therefore the equations (1) and (2) are identical.

By identifying the coefficients, we get

$$\frac{p_0}{p_n} = \frac{p_1}{p_{n-1}} = ... = \frac{p_{n-1}}{p_1} = \frac{p_n}{p_0} = k(say)$$

$$k^2 = \frac{p_0}{p_n} . \frac{p_n}{p_0} = 1$$

$$k = \pm 1$$

TYPE (i) : $k = 1$

We have $p_0 = p_n, p_1 = p_{n-1}, ...$

In this case the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and have the same sign.

TYPE (ii): $k = -1$

We have $p_0 = -p_n, p_1 = -p_{n-1}, ...$

In this case the coefficients of the terms equidistant from the beginning and the end are equal in magnitude but different in sign.

4.3.2 PROPERTIES OF THE RECIPROCAL EQUATIONS

Let $f(x) = p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + ...+ p_{n-1} x + p_n = 0 --- (1)$ be a reciprocal equation.

1. A reciprocal equation of the **TYPE(i)**and of odd degree has one of its roots equal to -1. Hence $x+1$ is a factor of $f(x)$.
2. A reciprocal equation of the **TYPE (ii)** and of even degree has two of its roots equal to +1 and -1. The coefficient of the middle term of such an equation is zero. Hence x^2-1 is a factor of $f(x)$.
3. A reciprocal equation of the **TYPE (ii)** and of odd degree has one of its roots equal to +1. Hence $x-1$ is a factor of $f(x)$.
4. As per properties (1), (2) and (3) after removing the factors, we will be left with a reciprocal equation of **TYPE (i)** and of an even degree.

The properties as mentioned above leads us for the following definition.

4.3.3 DEFINITION

Every reciprocal equation can be reduced to a reciprocal equation of **TYPE (i)** and of an even degree. This equation is called as standard reciprocal equation.

4.3.4 PROCEDURE FOR SOLVING A STANDARD RECIPROCAL EQUATION

Let the standard reciprocal equation as follows,

$$p_0x^{2n}+p_1x^{2n-1}+p_2x^{n-2}+.....+p_{n-1}x+p_n=0----(2)$$

Dividing by x^n, we have

$$p_0\left(x^n+\frac{1}{x^n}\right)+p_1\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+p_2\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+.....+p_n=0----(3)$$

Then substitute

$$x+\frac{1}{x}=y$$

$$x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=y^2-2$$

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3\left(x+\frac{1}{x}\right)=y^3-3y$$

$$x^n+\frac{1}{x^n}=\left(x^{n-1}+\frac{1}{x^{n-1}}\right)\left(x+\frac{1}{x}\right)-\left(x^{n-2}+\frac{1}{x^{n-2}}\right)$$

for all n, we can always reduce equation (3) to an equation of degree n in y.

(1) A RECIPROCAL EQUATION OF THE TYPE (I) AND OF ODD DEGREE

Example 1

Solve the equation $6x^5+11x^4-33x^3-33x^2+11x+6=0$

Solution

This is a reciprocal equation of the **TYPE(i)** and of odd degree has one of its roots equal to -1. Hence $x+1$ is a factor of $f(x)$.

-1 is a root of this equation

Reduce the equation by -1 using synthetic division

-1	6	11	-33	-33	11	6
0	-6	-5	38	-5	-6	
6	**5**	**-38**	**5**	**6**	**0**	

The reduced equation is $6x^4+5x^3-38x^2+5x+6=0$
To find the other roots of this equation, dividing by x^2,

$$6\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)-38=0$$

Put $x+\frac{1}{x}=y$ and $x^2+\frac{1}{x^2}=y^2-2$

We will be left with a reciprocal equation of **TYPE (i)** and of an even degree.

$$6(y^2-2)+5y-38=0$$

$$6y^2+5y-50=0$$

$$(3y+10)(2y-5)=0$$

$$y=-\frac{10}{3},\frac{5}{2}$$

$x+\frac{1}{x}=-\frac{10}{3}$ $\qquad$ $x+\frac{1}{x}=\frac{5}{2}$

$3x^2+10x+3=0$ $\qquad$ $2x^2-5x+2=0$

$x=-3,-\frac{1}{3}$ $\qquad$ $x=2,\ \frac{1}{2}$

The roots of the given equation are $x=-1,-3,-\frac{1}{3},2,\frac{1}{2}$

(2) A RECIPROCAL EQUATION OF THE TYPE (II) AND OF EVEN DEGREE

Example 2

Solve the equation $3x^6+x^5-27x^4+27x^2-x-3=0$

Solution

This is a reciprocal equation of the **TYPE (ii)** and of even degree has two of its roots equal to +1 and -1. The coefficient of the middle term of such an equation is zero. Hence x^2-1 is a factor of $f(x)$.

± 1 are the roots of this equation

Reduce the equation by -1 and +1, using synthetic division

1	3	1	-27	0	27	-1	-3
0		3	4	-23	-23	4	3
-1	3	4	-23	-23	4	3	0
0		-3	-1	24	-1	-3	
3	1	-24	1	3	0		

The reduced equation is $3x^4+x^3-24x^2+x+3=0$
To find the other roots of this equation, dividing by x^2,

$$3\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-24=0$$

Put $x+\frac{1}{x}=y$ and $x^2+\frac{1}{x^2}=y^2-2$

We will be left with a reciprocal equation of **TYPE (i)** and of an even degree.

$$3(y^2-2)+y-24=0$$

$$3y^2-y-30=0$$

$$(y-3)(3y+10)=0$$

$$y=3,-\frac{10}{3}$$

$x+\frac{1}{x}=-\frac{10}{3}$ $\qquad$ $x+\frac{1}{x}=\frac{5}{2}$

$3x^2+10x+3=0$ $\qquad$ $x^2-3x+1=0$

$x=-3,-\frac{1}{3}$ $\qquad$ $x=\frac{3\pm\sqrt{5}}{2}$

The roots of the given equation are $x=-1,1,-3,-\frac{1}{3},\frac{3+\sqrt{5}}{2},\frac{3-\sqrt{5}}{2}$

(3). A RECIPROCAL EQUATION OF THE TYPE (II) AND OF ODD DEGREE

Example 3

Solve the equation $6x^5-x^4-43x^3+43x^2+x-6=0$

Solution

This is a reciprocal equation of the **TYPE (ii)** and of odd degree has one of its roots equal to +1. Hence $x-1$ is a factor of $f(x)$.

+1 is a root of this equation

Reduce the equation by +1 using synthetic division

1	6	-1	-43	43	1	-6
0	6	5	-38	5	6	
6	**5**	**-38**	**5**	**6**	**0**	

The reduced equation is $6x^4+5x^3-38x^2+5x+6=0$

To find the other roots of this equation, dividing by x^2,

$$6\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)-38=0$$

Put $x+\frac{1}{x}=y$ and $x^2+\frac{1}{x^2}=y^2-2$

We will be left with a reciprocal equation of **TYPE (i)** and of an even degree.

$$6(y^2-2)+5y\text{-}38=0$$
$$6y^2+5y\text{-}50=0$$
$$(3y+10)(2y-5)=0$$
$$y=-\frac{10}{3},\frac{5}{2}$$

$x+\frac{1}{x}=-\frac{10}{3}$ $\qquad$ $x+\frac{1}{x}=\frac{5}{2}$

$3x^2+10x+3=0$ $\qquad$ $2x^2-5x+2=0$

$x=-3,-\frac{1}{3}$ $\qquad$ $x=2,\ \frac{1}{2}$

The roots of the given equation are $x=1,-3,-\frac{1}{3},2,\frac{1}{2}$

(4) A RECIPROCAL EQUATION OF STANDARD TYPE

Example 4

Solve the equation $4x^4\text{-}20x^3+33x^2-20x+4=0$

Solution

This is a reciprocal equation of standard type

$4x^4\text{-}20x^3+33x^2-20x+4=0$

To find the roots of this equation, dividing by x^2,

$$4\left(x^2+\frac{1}{x^2}\right)-20\left(x+\frac{1}{x}\right)+33=0$$

Put $x+\frac{1}{x}=y$ and $x^2+\frac{1}{x^2}=y^2-2$

We will be left with a reciprocal equation of **TYPE (i)** and of an even degree.

$$4(y^2-2)-20y+33=0$$
$$4y^2-20y+25=0$$
$$(2y-5)^2=0$$
$$y=\frac{5}{2},\frac{5}{2}$$

$$x+\frac{1}{x}=\frac{5}{2}$$
$$2x^2-5x+2=0$$
$$x=2,\ \frac{1}{2}$$

The roots of the given equation are $x=2,\frac{1}{2},2,\frac{1}{2}$

PROBLEMS FOR PRACTICE

1.*Solve the equation* $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$ *answer*: $3 \pm 2\sqrt{2}, 2 \pm \sqrt{3}$

2.*Solve the equation* $2x^6 - 9x^5 + 10x^4 - 3x^3 + 10x^2 - 9x + 2 = 0$

answer: $x = 2, \frac{1}{2}, \frac{1 \pm \sqrt{3}}{2}, \frac{3 \pm \sqrt{5}}{2}$

3.*Solve the equation* $60x^4 - 736x^3 + 1433x^2 - 736x + 60 = 0$ *answer*: $\frac{3}{2}, \frac{2}{3}, 10, \frac{1}{10}$

4.*Solve the equation* $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$ answer: $x = 1, 1, \frac{1 \pm i\sqrt{3}}{2}$

5.*Solve the equation* $8x^5 - 22x^4 - 55x^3 + 55x^2 + 22x - 8 = 0$ *answer*: $4, \frac{1}{4}, 1, -2, -\frac{1}{2}$

6.*Solve the equation* $2x^6 - x^3 - 2x^2 - x + 2 = 0$ answer: $x = -1, -1, \pm i, \frac{3 \pm i\sqrt{7}}{2}$

7.*Solve the equation* $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$

answer: $x = -1, +1, -\frac{1}{3}, -3, \frac{3 \pm \sqrt{5}}{2}$

4.4 REMOVAL OF TERMS

4.4.1 To Increase or Decrease(Diminish) the roots of a given equation by a given quantity.

Let $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ be the roots of the nth degree equation $f(x) = 0$

Suppose we have an equation whose roots are $\alpha_1 - h, \alpha_2 - h, \alpha_3 - h, ...\alpha_n - h$

Let $f(x) = p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + ... + p_{n-1} x + p_n = 0 - - - (1)$

$y = \alpha_1 - h$

Let $y = x - h$ $\sin ce\ x = \alpha_1\ is\ a\ root\ of\ f(x) = 0$

$x = y + h$

The given equation becomes $A_0 x^n + A_1 x^{n-1} + A_2 x^{n-2} + ... + A_{n-1} x + A_n = 0 - - - (2)$

Let $y = x - h$. The above equation is transformed back into the equation

$A_0 (x-h)^n + A_1 (x-h)^{n-1} + A_2 (x-h)^{n-2} + ... + A_{n-1}(x-h) + A_n = 0 - - - (3)$

Equation (1) and (3) are identical. Comparing (1) and (3) we can easily determine the co efficients $A_0, A_1, A_2, ..., A_{n-1}, A_n$

It can be easily shown that the co-efficient $A_n, A_{n-1}, A_{n-2}, ..., A_1, A_0$ are obtained by dividing $f(y)$ by $y - h$, n times in succession. A_0 is the quotient obtained in the last division.

The successive co-efficients can be determined by the method of synthetic division.

Example 1

Find the quotient and remainder when $3x^3 + 8x^2 + 8x + 12$ *is divided by* $x - 4$

Solution

Using synthetic division, divide by the root 4

	3	8	8	12
4	0	12	80	352
	3	**20**	**88**	**364**

The quotient is $3x^3+8x^2+8x+12$ and the remainder is 364

Example 2

Diminish the roots of $x^4-5x^3+7x^2-4x+5$ *by* 2

Solution

The coefficients in the transformed equations are the remainders when the polynomial is divided by $x-2$ in succession.

The division of the polynomial by $x-2$ can be as follows

	1	-5	7	-4	5
2	0	2	-6	2	-4
	1	**-3**	**1**	**-2**	**1**
	0	**2**	**-2**	**-2**	
	1	**-1**	**-1**	**-4**	
	0	**2**	**2**		
	1	**1**	**1**		
	0	**2**			
	1	**3**			

The transformed equation is

$x^4+3x^3+x^2-4x+1$

Example 3

Find the equation whose roots are of the equation

$x^4-x^3-10x^2+4x+24$ *incresed by* 2 *and hence solve the equation*

Solution

Diminish the root by -2

	1	-1	-10	4	24
-2	0	-2	6	8	-24
	1	**-3**	**-4**	**12**	**0**
	0	**-2**	**10**	**-12**	
	1	**-5**	**6**	**0**	
	0	**-2**	**14**		
	1	**-7**	**20**		
	0	**- 2**			
	1	**-9**			

The transformed equation is $x^4-9x^3+20x^2=0$

$x^2(x-4)(x-5)=0$

Therefore the roots are 0, 0, 4, 5.

The roots of the given equation are 0 - 2, 0 - 2, 4 - 2, 5 - 2 = -2, -2, 2, 3

Example 4

Transform the equation $x^4-4x^3-18x^2-3x+2=0$ into an equation with the thi rd term absent.

Solution

Let us Diminish the root by h so that the third term is removed

Let $y = x - h$

$x = y + h$

The given equation becomes

$$(y+h)^4 - 4(y+h)^3 - 18(y+h)^2 - 3(y+h) + 2 = 0.$$

$$y^4 + y^3(4h-4) + y^2(6h^2 - 12h - 18) + ... = 0$$

For the third term is to be absent

$$6h^2 - 12h - 18 = 0$$

$$h^2 - 2h - 3 = 0 \Rightarrow h = -1, 3$$

Diminish the root by -1

	1	-4	-18	-3	2
-1	0	-1	5	13	-10
	1	-5	13	10	-8
	0	-1	6	7	
	1	-6	-7	17	
	0	-1	7		
	1	-7	0		
	0	-1			
	1	-8			

The transformed equation is $x^4 - 8x^3 + 17x - 8 = 0$

Diminish the root by 3

	1	-4	-18	-3	2
3	0	3	-3	-63	-198
	1	-1	-21	-66	-196
	0	3	6	-45	
	1	2	-15	-111	
	0	3	15		
	1	5	0		
	0	3			
	1	8			

The transformed equation is $x^4 + 8x^3 - 111x - 196 = 0$

Example 4

Re*move* the second term from *the equation* $x^3 - 6x^2 + 11x - 6 = 0.$

Solution

Let us Diminish the root by h so that the second term is removed

Let $y = x - h$, $x = y + h$

The given equation becomes

$$(y+h)^3 - 6(y+h)^2 + 11(y+h) - 6 = 0.$$

$$y^3 + y^2(3h-6) + y(3h^2 - 12h + 11) + h^3 - 6h^2 + 11h - 6 = 0$$

If the second term is to be absent

$3h-6=0 \Rightarrow h=2$

Diminish the root by 2

	1	-6	11	-6
2	0	2	-8	6
	1	**-4**	**3**	**0**
	0	**2**	**-4**	
	1	**-2**	**-1**	
	0	**2**		
	1	**0**		

The transformed equation is $x^3-x=0$

PROBLEMS FOR PRACTICE

1.*Find the quotient and remainder when* $2x^6+3x^5-15x^2+2x-4$ *is divided by* $x+5$

answer : *quotient* : $2x^5-7x^4+35x^3-175x^2+860x-4298$, *remainder* : 21486

2.*Diminish the roots of* $x^4+x^3-3x^2+2x-4$ *by* 2

answer : $x^4+9x^3+27x^2+34x+12$

3.*Diminish the roots of* $x^4-x^3-10x^2+4x+24$ *by* 2 *and hence solve the original equation. answer* : $x^4+7x^3+8x^2-16x=0, x=0,1,-4,-4$

given roots: -2,-2,2,3

4.*Find the equation whose roots are of the equation* $3x^4-7x^3-15x^2+x-2$ *incresed by* 7.

answer: $3x^4-x^3-720x^2-287x+4058=0$

5.*Find the equation whose roots are of the equation* $4x^4+32x^3+83x^2+76x+21=0$ *incresed by* 2 *and hence solve the equation*

answer : $4x^4-13x^2+9=0, x=\pm1,\pm\frac{3}{2}$, *solu* : $-3,-1,-\frac{7}{2},-\frac{1}{2}$.

6.*Remove* the second term from *the equation* $x^2-6x^2+10x-3=0$.

answer : $x^3-2x+1=0$

7.*Remove* the second term from *the equation* $x^5+5x^4+3x^3+x^2+x+1=0$.

answer : $x^5-7x^3+12x^2-7x+2=0$

8.*Transform the equation* $x^4-4x^3-18x^2-3x+2=0$ into an equation with the t hird term absent. answer: $x^4-8x^3+17x-8=0$ (*or*) $x^4+8x^3-111x-196=0$

4.5 TRANSFORMATION IN GENERAL

Let $\alpha_1,\alpha_2,\alpha_3,......\alpha_n$ be the roots of the nth degree equation $f(x)=0$, it is required to find an equation whose roots are $\phi(\alpha_1),\phi(\alpha_2),\phi(\alpha_3),......\phi(\alpha_n)$. The relation between a root x of $f(x)=0$ and a root y of the required equation is $y=\phi(x)$.

If x be eliminated between $f(x)=0$ and , an equation in y is obtained which is the required equation.

Example 1

If α, β, γ *are the roots of the equation* $x^3 + px^2 + qx + r \equiv 0$, *form the equation whose roots are* $\alpha - \frac{1}{\beta\gamma}, \beta - \frac{1}{\gamma\alpha}, \gamma - \frac{1}{\alpha\beta}$

Solution

$$\alpha - \frac{1}{\beta\gamma} = \alpha - \frac{\alpha}{\alpha\beta\gamma} = \alpha - \frac{\alpha}{-r} \sin ce\ \alpha\beta\gamma = -r$$

We have $= \alpha + \frac{\alpha}{r}$

$$y = x + \frac{x}{r}$$

Therefore the required equation is obtained by eliminating x between the equations

$y = x + \frac{x}{r} ------ (1)$

$x^3 + px^2 + qx + r = 0 ---- (2)$

From (1), we get $x = \frac{yr}{1+r}$

Substituting this value of x in the equation (2), we get

$r^2y^3 + pr(1+r)y^2 + q(1+r)^2 y + (1+r)^3 = 0$

Example 2

If a, b, c *be the roots of the equation* $x^3 + px^2 + qx + r \equiv 0$, *form th e equation whose roots are* $bc - a^2, ca - b^2, ab - c^2$.

Solution

$$bc - a^2 = \frac{abc}{a} - a^2$$

We have

$$= -\frac{r}{a} - a^2 \sin ce \text{ abc = - r}$$

Hence the required equation is obtained by eliminating x between the equations

$y = -\frac{r}{x} - x^2 ---- (1)$

And $x^3 + px^2 + qx + r \equiv 0 ---- (2)$

From (1), we get $x^3 + xy + r \equiv 0 ---- (3)$

Subtracting (3) from (2), we get

$px^2 + qx - xy = 0$

$x(px + q - y) = 0$

$x = 0$ or $(px + q - y) = 0$

X cannot be equal to zero, therefore $(px + q - y) = 0$

$x = \frac{y-q}{p}$ substituting this value of x in equation (2), we get

$$\left(\frac{y-q}{p}\right)^3+p\left(\frac{y-q}{p}\right)^2+q\left(\frac{y-q}{p}\right)+r=0$$

$$y^3+(p^2-3q)y^2+(3q^2-p^2q)y+p^3r-q^3=0$$

Example 3

If $\alpha,\ \beta,\ \gamma$ *are the roots of the equation* $x^3-6x+7=0$, *form the equation whose roots are* $\alpha^2+2\alpha+3, \beta^2+2\beta+3, \gamma^2+2\gamma+3.$

Solution

Here we have eliminate x between the equations $x^3-6x+7=0$ ------(1)

and $y=x^2+2x+3$

$x^2+2x+(3-y)=0---(2)$

Multiplying (2) by x and subtracting (1) from it, we get

$2x^2+(9-y)x-7=0---(3)$

From (2) and (3), we get

$$\frac{x^2}{-14-(9-y)(3-y)}=\frac{x}{7+2(3-y)}=\frac{1}{(9-y)-4}$$

So that $(13-2\text{y})^2=(5-y)(-y^2+12y-41)$

$$y^3-21y^2+153y-374=0$$

PROBLEMS FOR PRACTICE

1.*If* $\alpha,\ \beta,\ \gamma$ *are the roots of the equation* $x^3+px^2+qx+r=0$, *find the* value of $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)$.*answer*: $(q-1)^2+(p-r)^2$

2.*If* $\alpha,\ \beta,\ \gamma$ *are the roots of the equation* $x^3+px^2+qx+r=0$, *find the* equation whose roots a re $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$.*answer*: $y^3+2py^2+(p^2+q)y+(pq-r)=0$

3.*If* $\alpha,\ \beta,\ \gamma$ *are the roots of the equation* $x^3+px^2+qx+r\equiv 0$, *form the e quation whose roots are*

(i)$\alpha+\frac{1}{\beta\gamma},\beta+\frac{1}{\gamma\alpha},\gamma+\frac{1}{\alpha\beta}$ *answer*: $r^2x^3+rp(1-r)x^2+q(1-r)^2x-(1-r)^3$

$(ii)\alpha^2,\beta^2,\gamma^2$ *answer* : $y^3+(2q-p^2)y^2+(q^2-2pr)y-r^2=0$

$(iii)\alpha^2+\alpha,\beta^2+\beta,\gamma^2+\gamma$

answer : $y^3+(2q+p-p^2)y^2+(q^2+q+3r-pq-2pr)y-r(q+1-p-r)=0$

$(iv)\alpha(\beta+\gamma),\beta(\gamma+\alpha),\gamma(\alpha+\beta)$.*answer* : $y^3+(2q)y^2+(q^2+pr)y+r^2-pqr=0$

4.6 DESCARTE'S RULE OF SIGNS

Descartes' rule of signs is a method for looking at a polynomial equation and estimating the number of positive, negative, and imaginary solutions.

When a polynomial is written in descending order, a variation of sign occurs when the signs of consecutive terms change.

Example 1
Discuss the sign change of the equation using Descarte's rule of sign
$P(x) = 3x^5 — 7x^4 — 8x^3 — x^2 + 3x — 9$

$+ \quad - \quad - \quad - \quad + \quad -$

1 2 3

there are sign changes in going from the first to the second terms, from the fourth to the fifth terms, and from the fifth to the sixth terms. So there are three variations in sign for P(-x) after it is simplified:
$P(-x)=3(-x)^5-7(-x)^4-8(-x)^3-(-x)^2+3(-x)-9$
$=-3x^5-7x^4+8x^3-x^2-3x-9$

$- \quad - \quad + \quad - \quad - \quad -$

1 2

In the polynomial $p(-x)$ the signs of the terms change from the second to the third terms, and then the signs change again from the third to the fourth terms. So there are two variations in sign for $p(-x)$.

Descarte's Rule of Signs

If $p(x)=0$ is a polynomial equation with real coefficients, then the number of positive roots of the equation is either equal to the number of variations of sign of $p(x)$ or less than that by an even number. The number of negative roots of the equation is either equal to the number of variations in sign of $p(-x)$ or less than that by an even number.

Example 2

Discuss the possibilities for the roots to $2x^3 - 5x^2 - 6x + 4 = 0$.

Solution

The number of variations of sign $p(x) = 2x^3 - 5x^2 - 6x + 4 = 0$ is 2.

According to Descartes' rule, the number of positive roots is either 2 or 0.
Since
$P(-x)=2(-x)^5-5(-x)^4-6(-x)^3+4=-2x^3-5x^2+6x+4$,
There is only one variation of sign in $p(-x)$.
So there is exactly one negative root.
If only one negative root exits, then the other two roots must be positive or imaginary. The number of imaginary roots is determined by the number of positive and negative roots because the total number of roots must be three. The following table summarizes these two possibilities.

Example 3

Discuss the possibilities for the roots to $3x^4 - 5x^3 - x^2 - 8x + 4 = 0$.

Solution

The number of variations of sign $P(x) = 3x^4 - 5x^3 - x^2 - 8x + 4$ *is* 2.

According to Descartes' rule, the number of positive roots is either 2 or 0.

Since

$P(-x) = 3(-x)^4 - 5(-x)^3 - (-x)^2 - 8(-x) + 4 = 3x^4 \underline{+ 5x^3 - x^2 + 8x} + 4$,

There are two variations of sign in $P(-x)$.

So the number of negative roots is either two or zero.

Each line of the following table gives a possible distribution of the type of roots to the equation.

Number of positive roots	**Number of Negative roots**	**Number of Imaginary roots**
2	2	0
2	0	2
0	2	2
0	0	4

Descartes' rule of signs adds to our knowledge of the roots of an equation. It is especially helpful when the number of variations of sign is zero or one. If there are no variations in sign for $P(x)$, then there are no positive roots. if there is one variation of sign in $P(x)$, then we know that one positive root exists.

PROBLEMS FOR PRACTICE

1. Prove that the equation $x^3 + 2x + 3 = 0$ has one negative root and two imaginary roots.
Answer ; one $-ve$ real and two imaginary roots

2. Show that the equation $x^7 - 3x^4 + 2x^3 - 1 = 0$ has atleast four imaginary roots.
3. Show that the equation $x^4 + 3x - 1 = 0$ has two real and two imaginary roots.
4. Show that the equation $3x^5 - 2x^3 - 4x + 2 = 0$ has three real and two imaginary roots

5. Prove that the equation $x^9 + x^5 + x^4 + x^2 + 1 = 0$ has one real root which is negative and eight imaginary roots.
6. Show that the equation $x^5 - 6x^2 - 4x + 5 = 0$ has atleast two imaginary roots
7. Show that the equation $12x^7 - x^4 + 10x^3 - 28 = 0$ has atleast four imaginary roots
8. Show that the equation $x^6 + 3x^2 - 5x + 1 = 0$ has atleast four imaginary roots
9. Show that the equation $x^5 + x^3 - 2x^2 + x - 2 = 0$ has atleast two imaginary roots.
10. Show that the equation $x^4 + 2x^2 + 3x - a = 0$ $(a > 0)$ has atleast two imaginary roots

4.8 ROLLE'S THEOREM

Theorem

Between two consecutive real roots a and b of the equation $f(x) = 0$ where $f(x)$ is a polynomial, there lies at least one real root of the equation $f'(x) = 0$.

Proof

Let $f(x)$ be $(x-a)^m (x-b)^n \phi(x)$, where m and n are positive integers and $\phi(x)$ is not divisible by $(x-a)$ or by $(x-b)$. Since a and b are consecutive real roots a and b of the equation $f(x) = 0$, the sign of in the interval $a \le x \le b$ is either positive throughout or negative throughout, for if it changes its sign between a and b, then there is a root of $\phi(x) = 0$ that is of $f(x) = 0$ lying between a and b, which is contrary to the hypothesis that a and b are consecutive roots.

$$f'(x) = (x-a)^m n(x-b)^{n-1} \phi(x) + m(x-a)^{m-1}(x-b)^n \phi(x) + (x-a)^m (x-b)^n \phi'(x)$$
$$= (x-a)^{m-1}(x-b)^{n-1} \chi(x),$$

where $\chi(x) = \{m(x-b) + n(x-a)\}\phi(x) + (x-a)(x-b)\phi'(x)$.

$\chi(a) = m(a-b)\phi(a)$

$\chi(b) = n(b-a)\phi(b)$

$\chi(a)$ and $\chi(b)$ have different signs since $\phi(a)$ and $\phi(b)$ have the same sign.

$\chi(x) = 0$ has atleast one root between a and b.
Hence $f'(x) = 0$ has atleast one root between a and b.

SOME IMPORTANT RESULTS

1. If all the roots of $f(x) = 0$ are all real, then all the roots of $f'(x) = 0$ are also real
2. If all the roots of $f(x) = 0$ are real, then the roots of $f'(x) = 0$, $f''(x) = 0$, $f'''(x) = 0$ are real.
3. At the most only one real root of $f(x) = 0$ can lie between two consecutive roots of $f'(x) = 0$, that is the real roots of $f'(x) = 0$ separate those of $f(x) = 0$.
4. If $f'(x) = 0$ has r real roots, then $f(x) = 0$ cannot have more than (r +1) real roots.
5. $f(x) = 0$ has atleast as many imaginary roots as $f'(x) = 0$.

The extended real number system is the set of real numbers R, together with two symbols $+\infty$ and $-\infty$ which satisfied.

a) If $x \in R$, then $x + (+\infty) = +\infty$, $x + (-\infty) = -\infty$
$x - (+\infty) = -\infty$, $x - (-\infty) = +\infty$
$\frac{x}{+\infty} = \frac{x}{-\infty} = 0$
b) If $x \succ 0$, then $x(+\infty) = +\infty$, $x(-\infty) = -\infty$
c) If $x \prec 0$, then $x(+\infty) = -\infty$, $x(-\infty) = +\infty$
d) $(+\infty) + (+\infty) = (+\infty)(+\infty) = (-\infty)(-\infty) = +\infty$
e) $x \in R$, then $-\infty < x < +\infty$,

6. If $\infty, -\infty$ is undefined then $\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}$ is also undefined and $0.\infty = 0$.

Example 1

Find the nature of the roots of the equation $4x^3 - 21x^2 + 18x + 20 = 0$.

Solution

Let us consider the function $f(x) = 4x^3 - 21x^2 + 18x + 20 = 0$.

We have $f'(x) = 12x^2 - 42x + 18$.

$= 6(2x-1)(x-3)$

Hence the real roots of $f'(x)=0$ are $\frac{1}{2}$ and 3. So, the roots of $f(x)=0$, if any will be in the intervals between $-\infty$ and $\frac{1}{2}$, $\frac{1}{2}$ and 3, 3 and $+\infty$ respectively.

x	$-\infty$	$\frac{1}{2}$	3	$+\infty$
$f(x)$	-	+	-	+

Therefore $f(x)$ must vanish, once in each of the above intervals. Hence $f(x)=0$ has three real roots.

Example 2

Show that the equation $3x^4-8x^3-6x^2+24x-7=0$ *has one positive, one negative and two imaginary roots.*

Solution

Let $f(x)=3x^4-8x^3-6x^2+24x-7$

We have $f'(x)=12x^3-24x^2-12x+24$

$=12(x+1)(x-1)(x-2)$

The roots of $f'(x)=0$ are -1, +1, +2

x	$-\infty$	$\frac{1}{2}$	3	$+\infty$
$f(x)$	-	+	-	+

$f(x)=0$ has a real root lying between -1 and $-\infty$, one between -1 and +1 and two imaginary roots.

We know that $f(+1)=+ve,$, $f(0)=-ve$

Therefore the real root lying between -1 and +1 lies between 0 and +1. Hence it is a positive root. The other real root lies between -1 and $-\infty$ and so it is a negative root.

Example 3

Find the nature of the roots of the equation $2x^3-9x^2+12x+3=0$

Solution

Let $f(x) = 2x^3 - 9x^2 + 12x + 3 = 0$

We have $f'(x) = 6x^3 - 18x + 12$

$= (x-1)(x-2)$

The roots of $f'(x) = 0$ are +1, +2

x	$-\infty$ +1 +2 $+\infty$
$f(x)$	- + + +

$f(x) = 0$ has a real root lying between $-\infty$ and +1 and two imaginary roots.

We know that $f(+1) = +ve,$, $f(0) = -ve$

Therefore the real root lying between $-\infty$ and +1 lies between 0 and +1. Hence it is a positive root.

Example 4

Discuss the reality of the roots $x^4 + 4x^3 - 2x^2 - 12x + a = 0$ *for all re al values of* a

Solution

Let $f(x) = x^4 + 4x^3 - 2x^2 - 12x + a = 0$

We have $f'(x) = 4x^3 + 12x^2 - 4x - 12$

$= 4(x+1)(x-1)(x+3)$

The roots of $f'(x) = 0$ are -1, +1, -3

x	$-\infty$ -3 -1 +1 $+\infty$
$f(x)$	+ a-9 7+a a-9 +

If a – 9 is negative and 7+a is positive, the four roots of $f(x) = 0$ are real.

If -7 < a < 9, $f(x) = 0$ has four real roots

If a > 9, then $f(x) = 0$ is positive throughout and henc all the roots of $f(x) = 0$ are imaginary.

If a < -7, the signs of $f(x) = 0$ at $-\infty$, -3, -1, 1, $+\infty$ are respectively

+ve, -ve, -ve,-ve, +ve.

Hence $f(x)=0$ has two real roots and two imaginary roots.

PROBLEMS FOR PRACTICE

1.*Find the nature of the roots of the equation*

(i) $x^4+4x^3-20x^2+10=0$

$(ii) 4x^3-21x^2+18x+20=0$

2.*prove that all the roots of the equation* $x^3-18x+25=0$

3.*show that the equation* $f(x) = (x-a)^3+(x-b)^3+(x-c)^3=0$ *has one real and two imaginary roots.*

4.*Find the* $\lim$*its to the value of* c *in order that the roots of the equation* $x^3-18x+25=0$ *may all real.*

Chapter 5
Solution of numerical equations

Contents

5.0 AIMS AND OBJECTIVES

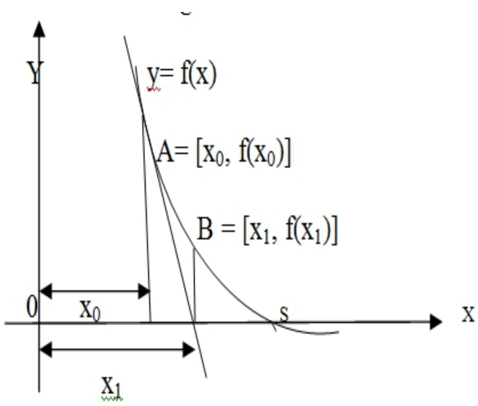
Our Aim is to learn about the **Solution of numerical equations**. Further we aim at learning Newton's method of approximation to a root, Horner's method and Strum's theorem.

5.1 SOLUTION OF NUMERICAL EQUATIONS

A polynomial equation, whose coefficients are numbers are called numerical equations. Such an equation may have real and imaginary roots. Among the real roots, some roots may be commensurable and some incommensurable. We shall give below some methods to determine the commensurable and approximations to incommensurable roots of a numerical equation.

5.2 NEWTON'S METHOD or NEWTON'S -RAPHSON METHOD

In 17^{th} century Newton discovered a method for estimating a solution or root of an algebraic equation by defining a sequence of numbers that become successively closer and closer to the root sought. His method is best illustrated graphically.



Let x be the actual root of the equation $f(x)=0$. Draw the graph of the curve $y=f(x)$ as shown in fig. let x_0 be chosen arbitrarily. We then locate the point $(x_0, f(x_0))$ on the curve and draw the tangent line to the curve at the point meeting the x- axis at x_1.

x_1 is the first approximation to s. we locate the point $(x_1, f(x_1))$ on the curve and draw the tangent line to the curve at the point meeting the x- axis at x_2.

x_2 is the first approximation to s. we locate the point $(x_2, f(x_2))$ on the curve and draw the tangent line to the curve at the point meeting the x- axis at x_3.

We then repeat the process to arrive as closely as possible to s. Now we will derive the formula to arrive at $x_0, x_1, x_2, x_3, \ldots\ldots$etc.

The slope of the tangent line at the point $(x_0, f(x_0))$ is $f'(x_0)$. $(x_1, 0)$ and are the two points on this tangent line.

The slope of the tangent line is $\frac{0-f(x_0)}{x_1-x_0}$ [using slope formula]

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Thus if we start with a given value x_0. We can obtain $x_0, x_1, x_2, x_3, \ldots\ldots$etc.

If any two consecutive values are equal and equal to s then s is one of the roots of $f(x)=0$.

5.2.1 THE CONDITION FOR CONVERGENCE OF NEWTON-RAPHSON METHOD

Newton's formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = \phi(x_n)$$

Which is similar to the iteration method?

Hence in anology to the convergence of iterative method, Newton-Raphson method converges when

$|\phi'(x)| < 1$

But $\phi'(x) = \frac{d}{dx}\left[x - \frac{f(x)}{f'(x)}\right]$

$= 1 - \left\{\frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}\right\}$

$= \left[\frac{f(x)f''(x)}{[f'(x)]^2}\right]$

Hence the condition for convergence of Newton-Raphson method is $\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$

$|f(x)f''(x)| < |[f'(x)]^2|$

This is the condition for convergence of Newton's Raphson method.

5.2.2 FINDING THE ROOTS OF ALGEBRAIC EQUATIONS USING NEWTON'S RAPHSON METHOD.

Example 1

Use Newton's Method to determine an approximation to the solution to $x^3 - 2x + 0.5 = 0$.
Find the approximation to four decimal places.

Solution

$f(x) = x^3 - 2x + 0.5$

$f'(x) = 3x^2 - 2$

$f(0) = 0.5(+ve)$

$f(1) = -1.5(-ve)$

Hence the root lies between 0 and 1. Since the value of $f(x)$ at $x = 0$ is very close to zero than the value of $f(x)$ at $x = 1$, we can say that the root is very close to 0. Therefore we can assume that $x_0 = 0$ is the initial approximation to the root.

The general formula for Newton's Method is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

The first approximation

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ here $x_0 = 0$

$x_1 = 0 - \frac{f(0)}{f'(0)} = -\frac{0.5}{-2} = \frac{0.5}{2} = 0.25$ $\quad$ $f(0) = 0.5$, $f'(0) = -2$

The second approximation

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ here $x_1 = 0.25$

$x_2 = 0.25 - \frac{f(0.25)}{f'(0.25)}$ $\quad$ $f(0.25) = (0.25)^3 - 2(0.25) + 0.5$, $f'(0.25) = 3(0.25)^2 - 2$

$= 0.2586$

The third approximation

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ here $x_2 =$ 0.2586

$x_3 = 0.2586 - \frac{f(0.2586)}{f'(0.2586)}$

$= 0.2586$

And now we've got two approximations that agree to 4 decimal places and so we can stop. We will assume that the solution is approximately x =0.2586

5.2.3 FINDING THE ROOTS OF TRANSCENDENTAL EQUATIONS USING NEWTON'S RAPHSON METHOD.

Example ***2***

Use Newton's Method to determine an approximation to the solution to $\cos x = x$ *that lies in the interval* $[0,2]$. *Find the approximation to six decimal places.*

Solution

$x_0 = 1$

$\cos x - x = 0$

$f(x) = \cos x - x$

$f'(x) = -\sin x - 1$

The general formula for Newton's Method.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

The first approximation

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ here $x_0 = 1$

$x_1 = 1 - \frac{f(1)}{f'(1)}$ $\quad$ $f(1) = \cos 1 - 1$ $\quad$ $f'(1) = -\sin 1 - 1$

$= 0.75036$

The second approximation

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

here $x_1 = 0.75036$

$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{f(0.75036)}{f'(0.75036)}$ $\quad$ $f(0.75036) = \cos(0.75036) - 0.75036$

$f'(0.75036) = -\sin(0.75036) - 1$

$= 0.73911$

The third approximation

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ here $x_2 = 0.73911$

$x_3 = 0.73911 - \frac{f(0.73911)}{f'(0.73911)}$ $\quad$ $f(0.73911) = \cos(0.73911) - 0.73911$

$f'(0.73911) = -\sin(0.73911) - 1$

$= 0.739085$

The forth approximation

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$ here x_3 =0.739085

$x_4 = 0.739085 - \frac{f(0.739085)}{f'(0.739085)}$ $\quad$ f(0.739085)=cos(0.739085)− 0.739085 $\quad$ f'(0.739085)=−sin(0.739085)−1

= 0.739085

And now we've got two approximations that agree to 6 decimal places and so we can stop. We will assume that the solution is approximately =0.739085

Example 3

Find a root of $x\log_{10} x - 1.2 = 0$ *by Newton Raphson method correct to three decimal places.*

Solution

$f(x) = x\log_{10} x - 1.2$

$f'(x) = \log_{10} x + x\frac{1}{x}\log_{10} e = \log_{10} x + 0.4343$

$f(2) = 2\log_{10}(2) - 1.2 = 2(0.3010) - 1.2 = -0.5980(-ve)$

$f(3) = 3\log_{10}(3) - 1.2 = 3(0.4771) - 1.2 = 0.2313(+ve)$

Here f(2) and f(3) are opposite in sign, therefore the root of f(x) = 0 lies between 2 and 3. Also we have the magnitude of f(3) is less than f(2). Therefore we can take the initial approximation to the root is

$x_0 = 3$

The general formula for Newton's Method. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

The first approximation

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ here x_0 = 3

$x_1 = 3 - \frac{f(3)}{f'(3)}$ $\quad$ $f(3) = 3\log_{10} 3 - 1.2$ $\quad$ $f'(3) = \log_{10} 3 + 0.4343$

= 2.746

The second approximation

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ here x_1 = 2.746

$x_2 = 2.746 - \frac{f(2.746)}{f'(2.746)}$ $\quad$ $f(2.746) = 2.746\log_{10}(2.746) - 1.2$ $\quad$ $f'(2.746) = \log_{10}(2.746) + 0.4343$

= 2.741

The third approximation

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ here x_2 = 2.741

$x_3 = 2.741 - \frac{f(2.741)}{f'(2.741)}$ $\quad f(2.741) = 2.741\log_{10}(2.741) - 1.2$

$\quad f'(2.741) = \log_{10}(2.741) + 0.4343$

$= 2.741$

And now we've got two approximations that agree to 3 decimal places and so we can stop. We will assume that the solution is approximately $= 2.741$

Example 4

Use $x_0 = 1$ *to find the approximation to the solution to* $\sqrt[3]{x} = 0$

Solution

Clearly the solution is x = 0,

but it does make a very important.Let's get the general formula for Newton's method

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = x_n - 3x_n = -2x_n$$

In fact we don't really need to do any computations here. These computations get farther and farther away from the solution, =0, with each iteration. Here are a couple of computations to make the point. $x_1 = -2,\ x_2 = 4, x_3 = -8,\ x_4 = 16$ etc.so,
in this case the method fails and fails spectacularly.

Example 5

Write down Newton's formula to find the square root of a positive number k .

Solution

$let\ \ x = \sqrt{k} \Rightarrow x^2 = k$

$f(x) = x^2 - k$

$f'(x) = 2x$

The general formula for Newton's Method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{(x_n^2 - k)}{2x_n}\right) = \frac{2x_n^2 - x_n^2 + k}{2x_n} = \frac{x_n^2 + k}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{k}{x_n}\right]$$

Which is the required formula to find the square root of a positive number k.

Example 6

Evaluate $\sqrt{5}$ *to four decimal places by Newton's formula.*

Solution

$let\ \ x = \sqrt{5} \Rightarrow x^2 = 5$

$f(x) = x^2 - 5$

$f(2) = (2)^2 - 5 = -1 = -ve$

$f(3) = (3)^2 - 5 = 4 = +ve$

Here f(2) and f(3) are opposite in sign, therefore the root of f(x) = 0 lies between 2 and 3. Also we have the magnitude of f(2) is less than f(3). Therefore we can take the initial approximation to the root is

$x_0 = 2$

The formula to find the square root of a positive number k is

$x_{n+1} = \frac{1}{2}\left[x_n + \frac{k}{x_n}\right]$ here k = 5

The first approximation

$x_1 = \frac{1}{2}\left[x_0 + \frac{5}{x_0}\right]$ here $x_0 = 2$

$= \frac{1}{2}\left[2 + \frac{5}{2}\right] = 2.25$

The second approximation

$x_2 = \frac{1}{2}\left[x_1 + \frac{5}{x_1}\right]$ here $x_1 = 2.25$

$= \frac{1}{2}\left[2.25 + \frac{5}{2.25}\right] = 2.2361$

The third approximation

$x_3 = \frac{1}{2}\left[x_2 + \frac{5}{x_2}\right]$ here $x_2 = 2.2361$

$= \frac{1}{2}\left[2.2361 + \frac{5}{2.2361}\right] = 2.2361$

And now we've got two approximations that agree to 4 decimal places and so we can stop. We will assume that the solution is approximately =2.2361

Example 7

Find an iterative formula to find the reciprocal of a given number N and hence find the value of $\frac{1}{19}$.

Solution

$let\ \ x = \frac{1}{N} \Rightarrow N = \frac{1}{x}$

$f(x)=\frac{1}{x}-N$

$f'(x)=-\frac{1}{x^2}$

The general formula for Newton's Method.

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\left(\frac{(\frac{1}{x_n}-N)}{-\frac{1}{x_n^2}}\right)=x_n-x_n^2\left((\frac{1}{x_n}-N)\right)=2x_n-Nx_n^2=x_n(2-Nx_n)$$

$x_{n+1}=x_n(2-Nx_n)$ is the iterative formula.

To find $\frac{1}{19}$, take N = 19

Further $\frac{1}{20}=0.05$ take $\alpha_0=0.05$

$x_1=0.05(2-19\times 0.05)=0.0525$

$x_2=0.0525(2-19\times 0.0525)=0.05263125$

$x_3=0.05263125(2-19\times 0.05263125)=0.0526315789$

$x_4=0.0526315789$

Hence the value of $\frac{1}{19}$ is 0.0526315789

PROBLEMS FOR PRACTICE

1.*What is the order of convergence of Newton – Raphson method if the multiplicity of the root is one.answer* : *Order of convergence of Newton's Raphson method is* 2

2.*Newton – Raphson method is also known as the method of – – – –*

answer : Newton's iteration method.

3.*What is the rate of convergence in N.R method*?

answer :*The rate of convergence in N.R method is of order* 2.

4.*Write the iterative formula of N.R method*.

answer : $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

5.*What are the merits of the N.R method*?

answer : *Newton's method is successfully used to improve the result obtained by other methods. It is applicable to the solution of equations involving algebraical functions as well as transcendental functions.*

6.*Find a root of the given equation by Newton Raphson method correct to three decimal places.*

i) $2x^3-3x-6=0$. *answer* : 1.783769(4^{th} *iteration*)

ii) $x^3-6x+4=0$. *answer* : 0.73205081(4^{th} *iteration*)

iii) $3x-\cos x-1=0$. *answer* : 0.607102(3^{rd} *iteration*)

$iv) 4x - e^{x} = 0. answer: 2.1533(4^{th}\ iteration)$

$7.$ *Evaluate* $\sqrt{12}$ *to four decimal places by Newton's formula.*

answer $: 3.4641$

5.3 HORNER'S METHOD

This method is most elegant method to find the real positive or negative roots of a numerical equations

5.3.1 PROCEDURE FOR HORNER'S METHOD

Suppose a positive root of $f(x) = 0$ lies between α and $\alpha + 1$ (where α is an integer)

Step1: let that root be $\alpha . \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4 \ldots\ldots\ldots$

Step2: Diminish the roots of $f(x) = 0$ by the integral part a and let $\phi_1(x) = 0$ possess the root $0 . \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4 \ldots\ldots\ldots$

Step3: Multiply the roots of $\phi_1(x) = 0$ by 10 and let $\phi_2(x) = 0$ possess the root $\alpha_1 . \alpha_2\ \alpha_3\ \alpha_4 \ldots\ldots\ldots$

Step4: Find the value of α_1 and then diminish the roots by α_1 and let $\phi_3(x) = 0$ possess a root $0.\ \alpha_2\ \alpha_3\ \alpha_4 \ldots\ldots$

Step5: Repeating the process we find $\alpha_2, \alpha_3, \alpha_4, \ldots\ldots$ each time.

Example 1

Find the positive root of $x^3 + 3x - 1 = 0$, *correct to two decimal places, by Horner's method.*

Solution

Let $f(x) = x^3 + 3x - 1 = 0$

$f(0) = -ve; f(1) = +ve$. The positive root lies between 0 and 1.

Step1: Let it be $0.\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4 \ldots\ldots$

Step2: since the integral part is zero, diminishing the root by the integral part is not necessary.

Step3: multiply the roots by 10.

$\phi_1(x) = x^3 + 300x - 1000 = 0$ has root $\alpha_1 . \alpha_2\ \alpha_3\ \alpha_4 \ldots\ldots$

$\phi_1(3) = -ve; \phi_1(4) = +ve,$ therefore $\alpha_1 = 3.$

Now the root is $3. \alpha_2\ \alpha_3\ \alpha_4 \ldots$

Step4: Diminish root of $\phi_1(x) = 0$ by 3.

3	1	0	300	-1000
	0	3	9	927
3	1	3	309	-73
	0	3	18	
3	1	6	327	
	0	3		
	1	9		

$\phi_2(x) = x^3 + 9x^2 + 327x - 73 = 0$ has the root of $0.\alpha_2\,\alpha_3\,\alpha_4\ldots\ldots$

Repeat Step3: multiply the roots by 10.

$\phi_3(x) = x^3 + 90x^2 + 32700x - 73000 = 0$ has root $\alpha_2\,.\,\alpha_3\,\alpha_4\ldots\ldots$

$\phi_3(2) = -ve; \phi_3(3) = +ve,$ therefore $\alpha_2 = 2.$

Now the root is $2.\,\alpha_3\,\alpha_4\ldots$

Step4: Diminish root of $\phi_3(x) = 0$ by 2.

2	1	90	32700	-73000
	0	2	184	65768
2	1	92	32884	-7232
	0	2	188	
2	1	94	33072	
	0	2		
	1	96		

$\phi_4(x) = x^3 + 96x^2 + 33072x - 7232 = 0$ has the root of $0.\,\alpha_3\,\alpha_4\ldots\ldots$

Repeat Step3: multiply the roots $\phi_4(x) = 0$ by 10.

$\phi_5(x) = x^3 + 960x^2 + 3307200x - 7232000 = 0$ has root $\alpha_3\,.\,\alpha_4\ldots\ldots$

$\phi_5(2) = -ve; \phi_5(3) = +ve,$ therefore $\alpha_3 = 2.$

Hence the root is 0.322……..

Therefore the root, correct to two decimal places is 0.32.

Example 2

Find the positive root between 1 *and* 2 *which satisfies* $x^3 - 3x + 1 = 0,$ *correct to three decimal places, by Horner's method.*

Solution

Let $f(x) = x^3 - 3x + 1 = 0$

$f(1) = -ve; f(2) = +ve$. The positive root lies between 0 and 1.

Step1: Let it be $1.\,\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\ldots\ldots$

Step2: diminishing the root by the integral part 1.

1	1	0	-3	1
	0	1	1	-2
1	1	1	-2	-1
	0	1	2	
1	1	2	0	
	0	1		
	1	3		

$\phi_1(x) = x^3 + 3x^2 - 1 = 0$ has root $\alpha_1\,.\,\alpha_2\,\alpha_3\,\alpha_4\ldots\ldots$

$\phi_1(5) = -ve; \phi_1(6) = +ve,$ therefore $\alpha_1 = 5.$

Now the root is $5.\ \alpha_2\ \alpha_3\ \alpha_4 \ldots$

Step3: multiply the roots by 10.

$\phi_2(x) = x^3 + 300x^2 - 1000 = 0$

Step4: Diminish root of $\phi_2(x) = 0$ by 5.

5	1	300	0	-1000
	0	5	175	875
5	1	35	175	-125
	0	5	200	
5	1	40	375	
	0	5		
	1	45		

$\phi_3(x) = x^3 + 45x^2 + 375x - 125 = 0$ has the root of $0.\alpha_2\ \alpha_3\ \alpha_4 \ldots\ldots$

Repeat Step3: multiply the roots by 10.

$\phi_4(x) = x^3 + 450x^2 + 37500x - 125000 = 0$ has root $\alpha_2 . \alpha_3\ \alpha_4 \ldots\ldots$

$\phi_4(3) = -ve; \phi_4(4) = +ve,$ therefore $\alpha_2 = 3.$

Now the root is $3.\ \alpha_3\ \alpha_4 \ldots$

Step4: Diminish root of $\phi_3(x) = 0$ by 3.

3	1	450	37500	-125000
	0	3	1359	116577
3	1	453	38859	-8423
	0	3	1368	
3	1	456	40227	
	0	3		
	1	459		

$\phi_5(x) = x^3 + 459x^2 + 40227x - 8423 = 0$ has the root of $0.\ \alpha_3\ \alpha_4 \ldots\ldots$

Repeat Step3: multiply the roots $\phi_5(x) = 0$ by 10.

$\phi_6(x) = x^3 + 4590x^2 + 4022700x - 8423000 = 0$ has root $\alpha_3 . \alpha_4 \ldots\ldots$

$\phi_6(2) = -ve; \phi_6(3) = +ve,$ therefore $\alpha_3 = 2.$

Step4: Diminish root of $\phi_3(x) = 0$ by 2.

2	1	4590	4022700	-8423000
	0	2	1359	8063768
2	1	4592	4031884	-359232
	0	2	9188	
2	1	4594	4041072	
	0	2		
	1	4596		

$\phi_7(x) = x^3 + 4596x^2 + 4041072x - 359232 = 0$ has the root of $0.\ \alpha_4\ \alpha_5 \ldots\ldots$
Hence the root is $1.532\ldots\ldots$

5.3.2 Note :
In the above problem they mention to find the root between 1 and 2. Suppose the root interval not given in the problem, find the root as per the example 1 in chapter 5.3.1., (or using Rolle's theorem) we will have the root between 0 and 1. Here the answer is 0.376.

PROBLEMS FOR PRACTICE

1. *A sphere of pine wood, 2 metres in diameter, floating in water sinks to the depth of h metre, given by* $h^3 - 3h^2 + 2.5 = 0$, *Find the value of h correct to two decimal places by Horner's method.*

2. *Solve for the root specified below, by Horner's method.*

(i) $x^3 - 2x^2 - 3x - 4 = 0$ whose roots lies between 3 and 4 answer:3.2624

(ii) $x^3 - 8x - 40 = 0$ (*positive root*) *answer*: 3.284

(iii) $x^3 + 3x^2 - 12x - 11 = 0$ (*positive root*) answer:2.7689

(iv) $x^3 - 6x - 18 = 0$ (*positive root*) answer:3.18

(v) $x^3 - x = 9$ (*positive root*) answer:2.24

(vi) $10x^3 - 15x + 3 = 0$ whose roots lies between 1 and 2 answer:1.11

5.4 STRUM'S THEOREM

Let $p(x) = 0$ be an equation having no equal roots. let $p_1(x)$ be the first derived function of $p(x)$. let the process of finding the greatest common measure of $p(x)$ and $p_1(x)$ be performed. Let q_1 be the quotient and $p_2(x)$ the remainder with the sign changed.

$$p(x) = q_1\, p_1(x) - p_2(x)$$

Similar operation can be performed between $p_1(x)$ and $p_2(x)$

and we get $\quad p_1(x) = q_2\, p_2(x) - p_3(x)$

if we continue the operations, we get

$$p_2(x) = q_3\, p_3(x) - p_4(x)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$p_{r-1}(x) = q_r\, p_r(x) - p_{r+1}(x) \text{ and so on.}$$

The successive remainders with their signs changed, i.e., the functions $p_2(x)$, $p_3(x)$, ……will go on diminishing in degree till we reach a numerical remainder, say $p_n(x)$.

These functions $p(x)$, $p_1(x)$, $p_2(x)$,……… $p_n(x)$ are called **strum chain** or **strum's functions.**

5.4.1 Strum's function

The difference between the number of changes of sign in the series of strum's functions when a is substituted for x and the number when b is substitute for x express exactly the number of real roots of the equation $p(x)=0$ between a and b. This is known as strum's theorem and proof of this theorem is beyond the scope of this book.

Substitute 0 and ∞ in the series of strum's functions and the difference between the number of changes of sign will give the **positive roots.**

Substitute $-\infty$ and 0 in the series of strum's functions and the difference between the number of changes of sign will give the **negative roots.**

5.4.2 Note

The difference between the number of changes of sign when $-\infty$ and ∞ are substituted in the series of strum's function will give the number of real roots of the equation.

5.4.3 Procedure for finding the number of real roots using strum's function

Step 1: Let the given function $p(x) = 0$

Step2: Find $p_1(x)$, the first derivative of $p(x)$, i.e., find $p_1(x) = p'(x)$

Step3: Find $p_2(x)$ be the remainder of $\frac{p(x)}{p_1(x)}$ with the sign changed.

Step4: Find $p_3(x)$ be the remainder of $\frac{p_1(x)}{p_2(x)}$ with the sign changed.

Step5:Continue the operations, the successive remainders with their sings changed. i.e., find $p_1(x)$, $p_2(x)$, $p_3(x)$….. will go on diminishing in degree till we reach a numerical remainder.

Step6:Tabulate the signs of strum's functions after $-\infty$, 0 and ∞ are substituted in it.i.e., the signs of $p(-\infty)$, $p(0)$ and $p(\infty)$.

Step7:Calculate the number of changes of sign in step6, it will give the number of real roots of the equation given.

Step8: Difference between the number of changes of sign in $p(0)$ and $p(\infty)$ gives the number of Positive roots.

Step9: Difference between the number of changes of sign in $p(-\infty)$ and $p(0)$ gives the number of Negative roots.

Example 1

Find the number of real root of the equation $x^4-14x^2+16x+9=0$

Solution

$x^4-14x^2+16x+9=0$

$p(x)=x^4-14x^2+16x+9=0$

$p_1(x)=p'(x)=4x^3-28x+16$

$p_2(x)=$ - (The remainder of $\frac{p(x)}{p_1(x)}$)

Find $\dfrac{p(x)}{p_1(x)} = \dfrac{x^4-14x^2+16x+9}{4x^3-28x+16} = \dfrac{x^4-14x^2+16x+9}{x^3-7x+4}$

$$\begin{array}{r|l} & x \\ \hline x^3-7x+4 & x^4-14x^2+16x+9 \\ & x^4-7x^2+4x \\ \hline & -7x^2+12x+9 \end{array}$$

$p_2\left(x\right)= 7x^2-12x-9$

$p_3\left(x\right)= -$ (The remainder of $\dfrac{p_1(x)}{p_2(x)}$) $=$

Find $\dfrac{p_1(x)}{p_2(x)} = \dfrac{x^3-7x+4}{7x^2-12x-9} = \dfrac{x^3-7x+4}{x^2-\frac{12}{7}x-\frac{9}{7}}$

$$\begin{array}{r|l} & x+\frac{12}{7} \\ \hline x^2-\frac{12}{7}x-\frac{9}{7} & x^3+0x^2-7x+4 \\ & x^3-\frac{12}{7}x^2-\frac{9}{7}x \\ \hline & \frac{12}{7}x^2-\frac{40}{7}x+4 \\ & \frac{12}{7}x^2-\frac{144}{49}x-\frac{108}{49} \\ \hline & -\frac{136}{49}x+\frac{304}{49} \end{array}$$

$p_3\left(x\right) = -(-\frac{136}{49}x+\frac{304}{49}) = \frac{136}{49}x-\frac{304}{49} = 136x-304 = x-\frac{304}{136} = x-\frac{38}{17}$

$p_4\left(x\right)= -$ (The remainder of $\dfrac{p_2(x)}{p_3(x)}$)

Find $\dfrac{p_2(x)}{p_3(x)} = \dfrac{x^2-\frac{12}{7}x-\frac{9}{7}}{x-\frac{38}{17}}$

$$\begin{array}{r|l} & x+\frac{12}{7} \\ \hline x-\frac{38}{17} & x^2-\frac{12}{7}x-\frac{9}{7} \\ & x^2-\frac{38}{17}x \\ \hline & \end{array}$$

$$\begin{array}{r} \frac{62}{119}x - \frac{9}{7} + 4 \\ \frac{62}{119}x - \frac{2356}{2023} \\ \hline -\frac{1715}{14161} \end{array}$$

$$p_4(x) = -\frac{1715}{14161}$$

The sign of strum's function when $-\infty$, 0, and $+\infty$ are substituted are tabulated below:

x	$p(x)$	$p_1(x)$	$p_2(x)$	$p_3(x)$	$p_4(x)$	No.of changes of sign
$-\infty$	+ve	-ve	+ve	-ve	+ve	4
0	+ve	+ve	-ve	-ve	+ve	2
$+\infty$	+ve	+ve	+ve	+ve	+ve	0

Hence the number of real roots is 4, two of which are positive and the other two negative.

Example 2

Find the location and the number of real root of the equation $x^5 - 3x - 1 = 0$

Solution

$x^5 - 3x - 1 = 0$

$p(x) = x^5 - 3x - 1 = 0$

$p_1(x) = p'(x) = 5x^4 - 3$

$p_2(x) = \frac{1}{5}(12x + 5)$

$p_3(x) = \frac{59083}{20736}.$

The following table shows the signs of $p_n(x)$ and the number of sign changes

x	$p(x)$	$p_1(x)$	$p_2(x)$	$p_3(x)$	No.of changes of sign
-2	-ve	+ve	-ve	+ve	3
-1	+ve	+ve	-ve	+ve	2
0	-ve	-ve	+ve	+ve	1
1	-ve	+ve	+ve	+ve	1
2	+ve	+ve	+ve	+ve	0

This shows that 3-1 =2 real roots lie in (-2,0) and 1-0 =1 real root lies (0,2).
The equation has 3 real roots one between -2 and -1, one between -1 and 0 and one root between 1 and 2. The other two roots are complex roots.

Example 3

Determine the number of real roots of the equation $x^5 - ax - b = 0$ *when* $a, b > 0$ *and* $4^4 a^5 > 5^5 b^4$.

Solution

$x^5 - ax - b = 0$

$p(x) = x^5 - ax - b = 0$

$p_1(x) = p'(x) = 5x^4 - a$

$p_2(x) = \quad (4ax + 5b)$

$p_3(x) \quad = \quad 4^4 a^5 \text{ - } 5^5 b^4$.

The following table shows the signs of $p_n(x)$ and the number of sign changes

X	$p(x)$	$p_1(x)$	$p_2(x)$	$p_3(x)$	No.of changes of sign
-∞	-ve	+ve	-ve	+ve	3
0	-ve	-ve	+ve	+ve	1
∞	+ve	+ve	+ve	+ve	0

This shows that 3-1 =2 negative real roots lie in (-∞,0) and 1-0 =1 one positive real root lies (0, ∞). The equation has 3 real roots, the other two roots are complex roots.

Example 4

Deter min *e the number of real roots* in some range for *the equation* $x^4 - x^3 - x - 1 = 0$

Solution

$x^4 - x^3 - x - 1 = 0$

$p(x) = x^4 - x^3 - x - 1 = 0$

$p_1(x) = p'(x) = 4x^3 + 3x^2 - 1$

$p_2(x) = \quad \frac{3}{16}x^2 + \frac{3}{4}x + \frac{15}{16}$

$p_3(x) \quad = \quad -32x \text{ - } 64$.

$p_4(x) = \frac{-3}{16}$

The following table shows the signs of $p_n(x)$ and the number of sign changes

X	$p(x)$	$p_1(x)$	$p_2(x)$	$p_3(x)$	$p_4(x)$	No.of changes of sign
-∞	+ve	-ve	+ve	+ve	-ve	3
0	-ve	-ve	+ve	-ve	-ve	2
∞	+ve	+ve	+ve	-ve	-ve	1

This shows that 3-2 =1 negative real roots lie in (-∞,0) and 2-1 =1 one positive real root lies (0, ∞). The equation has 2 real roots, the other two roots are complex roots.

To Verify this result

$p(x) = x^4 - x^3 - x - 1 = 0$

can be factorized as

$(x^2 - 1)(x^2 + x + 1) \Rightarrow x = \pm 1,$ *are two real roots*

we know that $x^2 + x + 1$ *no real roots.*

Therefore it is verified.

5.4.4 The strum function satisfied the following conditions

1.Two neighbouring functions do not vanish simultaneously at any point in the interval.
2. At a null point of a strum function, its neighbouring functions are of different signs.
3. Within a sufficiently small interval surrounding a zero point of $p(x)$. $p_1(x)$ is everywhere greater than zero or everywhere smaller than zero.

PROBLEMS FOR PRACTICE

1.*Find the number of real root of the equation given*

$i.x^4 - 4x^3 - 4x - 13 = 0$ answer : 2 real roots

ii.$x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$ *answer* : 3*real*, 2*img*

2. *Find the condition that all the roots of the equation* $x^4 - 4x^3 - 4x - 13 = 0$ *may be real*.

answer : $4p^3 + 27q^2$ *must be negative.*

3.*Find the number and position of real roots of the equation* $x^3 - 3x + 6 = 0$

answer : 1 -*ve real lies between* -2 *and* -3.

note : *sub* -1, -2, -3 *until you get the same no.of changes of sign*(since *the root is negative*)

4. *Find the number of real roots in the following*

$i.x^4 - 14x^2 + 16x + 9 = 0$ *answer* : $+ve2, -ve2$

$ii.x^3 - 3x + 6 = 0$ *answer* : $2 + ve, 1 - ve$.

$iii.x^4 + 4x^3 - 4x - 13 = 0$ *answer* : $1 - ve, 3 = ve$.

Chapter 6
GENERAL SOLUTION OF THE CUBIC AND BIQUADRATIC EQUATIONS

Contents

6.0 AIMS AND OBJECTIVES

Our Aim is to learn what is meant by a cubic equation and how such an equation can be solved. The general strategy for solving a cubic equation is to reduce it to a quadratic equation, and then solve the quadratic by the usual means, either by factorizing or using the formula.

6.1 CUBIC EQUATIONS AND THE NATURE OF THEIR ROOTS

A cubic equation is of the form

$$ax^3 + bx^2 + cx + d = 0$$

It must have the term in x^3 or it would not be cubic (and so $a \neq 0$), but any or all of b, c and d can be zero.

For instance, $x^3 - 6x^2 + 11x - 6 = 0, 4x^3 + 57 = 0$, $x^3 + 9x = 0$ are all cubic equations.

Method 1

Let the cubic equation be $x^3 + ax + b = 0$

Case 1: The equation can be written as $x^3 = -ax - b$.

The x- coordinates of the intersection of the curves $y = x^3$ and $y = -ax - b$ will give the roots of the equation.

$y = x^3$ curve has a point of inflection at the origin.

Case 2: Multiply the equation by x.

We get $x^4 + ax + bx = 0$
$(x^2)^2 + x^2 + (a-1)x^2 + bx = 0$

We can easily see that the roots of the equation are the x- coordinates of the points of intersection of the parabola $y = x^2$ and the circle $y^2 + x^2 + (a-1)y + bx = 0$

Here the origin is to be excluded since we have multiplied the equation by x.

Method 2

If the cubic equation is $ax^3 + bx^2 + cx + d = 0$ **.** We can diminish the roots of the cubic by h and get an equation without the x^2 term. One of the above methods can be adopted to get the roots. The equation can be written as $ax^3 = -bx^2 - cx - d$

$$x^3 = -\frac{b}{a}x^2 - \frac{c}{a}x - \frac{d}{a}.$$

The roots are the x- coordinates of the intersection of the curves

$y = x^3$ and $y = -\frac{b}{a}x^2 - \frac{c}{a}x - \frac{d}{a}.$

Just as a quadratic equation may have two real roots, so a cubic equation has possibly three. But unlike a quadratic equation which may have no real solution, a cubic equation always has **at least one real root**. We will see why this is the case later. If a cubic does have three roots, two or even all three of them may be repeated. This gives us four possibilities which are illustrated in the following examples.

6.2 GRAPHICAL SOLUTIONS OF NUMERICAL EQUATIONS

To get real root of $f(x) = 0$ we draw the graph of the function $y = f(x)$ and find the x - coordinates of the points of intersection of the graph and the x - axis. It is not always easy to get exactly the shape of the curve when the degree of the curve is more than the second. To solve cubic and biquadratic equations the following method can be conveniently used. The method consists in constructing two convenient curves the x - coordinates of whose points of intersection are the roots of the given equation.

Example 1

Solve the equation $x^3 - 6x^2 + 11x - 6 = 0$

Solution

This equation can be factorized to give $(x-1)(x-2)(x-3) = 0$

This equation has three real roots, all different - the solutions are $x = 1$, $x = 2$ and $x = 3$.

In Figure 1 we show the graph of $y = x^3 - 6x^2 + 11x - 6.$

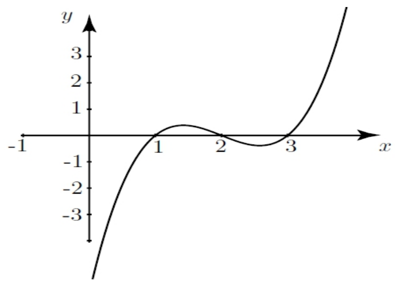


Figure 1. The graph of $y = x^3 - 6x^2 + 11x - 6$.

Notice that it starts low down on the left, because as x gets large and negative so does x^3 and it finishes higher to the right because as x gets large and positive so does x^3. The curve crosses the x-axis three times, once where $x = 1$, once where $x = 2$ and once where $x = 3$. This gives us our three separate solutions.

Example 2

Solve the equation $x^3 - 5x^2 + 8x - 4 = 0$.

Solution

This equation can be factorized to give $(x-1)(x-2)^2 = 0$

In this case we do have three real roots but two of them are the same because of the term $(x-2)^2$. So we only have two distinct solutions.

Figure 2 shows a graph of $y = x^3 - 5x^2 + 8x - 4$.

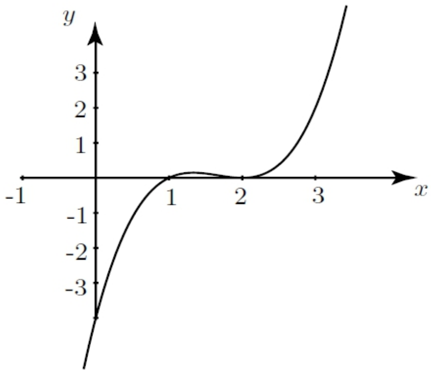


Figure 2. The graph of $y = x^3 - 5x^2 + 8x - 4$.

Again the curve starts low to the left and goes high to the right. It crosses the x-axis once and then just touches it at $x = 2$. So we have our two roots $x = 1$ and $x = 2$.

Example 3

Solve the equation $x^3 - 3x^2 + 3x - 1 = 0$.

Solution

This equation can be factorised to give $(x-1)^3 = 0$

So although there are three factors, they are all the same and we only have a single solution $x = 1$. The corresponding curve is $y = x^3 - 3x^2 + 3x - 1$ and is shown in Figure 3

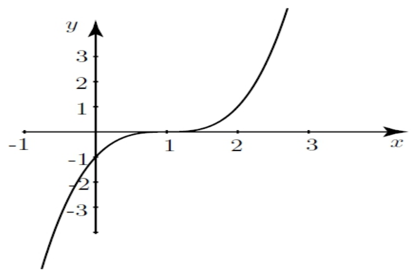


Figure 3. The graph of $y = x^3 - 3x^2 + 3x - 1$.

As with all the cubics we have seen so far, it starts low down on the left and goes high up to the right. Notice that the curve does cross the x-axis at the point $x = 1$ but the x-axis is also a tangent to the curve at this point. This is indicative of the fact that there are three repeated roots.

Example 4

Solve the equation $x^3 + x^2 + x - 3 = 0$.

Solution

This equation can be factorised to give $(x-1)\left(x^2 + 2x + 3\right) = 0$

The quadratic $x^2 + 2x + 3 = 0$ has no real solutions, so the only solution to the cubic equation is obtained by putting $x - 1 = 0$, giving the single real solution $x = 1$.The graph $y = x^3 + x^2 + x - 3$ is shown in Figure 4.

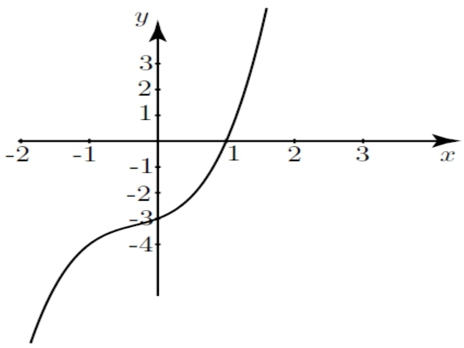


Figure 4. The graph of $y = x^3 + x^2 + x - 3$.

You can see that the graph crosses the x-axis in one place only. From these graphs you can see why a cubic equation always has at least one real root. The graph either starts large and negative and finishes large and positive (when the coefficient of x^3 is positive), or it will start large and positive and finish off large and negative (when the coefficient of x^3 is negative).

Note : The graph of a cubic **must** cross the x-axis at least once giving you at least one real root. So, any problem you get that involves solving a cubic equation will have a real solution.

PROBLEMS FOR PRACTICE

1. *Determine the real roots of the following cubic equations – if a root is repeated say how many times.*

$a)(x+1)(x-2)(x-3) = 0 \quad b)\ (x+1)(x^2-12x+20) = 0 \quad c)\ (x-3)^2(x+4) = 0$

$d)(x+2)(x^2+6x+10) = 0 \quad e)(x+2)(x^2+7x+10) = 0 \quad f)\ (x-5)(x^2-10x+25) = 0$

If you cannot find a solution by other methods then draw an accurate graph of the cubic expression. The points where it crosses the x –axis will give you the solutions to the equation. but their accuracy will be limited to the accuracy of your graph.

Example 1

Solve $x^3 + 4x^2 + x - 5 = 0.$

Solution

Now this equation will not yield a factor by any of the methods that we have discussed. So a graph of $y = x^3 + 4x^2 + x - 5$ has been drawn as shown in Figure 5.

No.	Name	Description	Value
1	Function f		f: $y = x^3$
2	Parabola c		c: $y = 4x^2 + x - 5$
3	Point A	Point on f	A = (-1.13, -1.45)
4	Point B	Point on f	B = (1.17, 1.62)
5	Point C	Point on f	C = (3.98, 62.84)

Figure 5. A graph of $y = x^3 + 4x^2 + x - 5$

It crosses the x-axis at three places and hence there are three real roots. As we stated before their accuracy will be limited to the accuracy of the graph. From the graph we find the approximate solutions $x \approx 3.98, 1.17, -1.13$.

Problems for practice

1.*Use graphs to solve, as accurately as you can, the following cubic equations.*

$a) x^3 - 4x^2 - 6x + 5 = 0$

$b)\ 2x^3 - 3x^2 - 4x - 35 = 0$

$c) x^3 - 3x^2 - x + 1 = 0$

2. *Find graphically all the roots of the following equations*

$a) x^3 - 7x + 6 = 0.$

By case 1, the equation can be written as $x^3 = 7x - 6$.

The x - coordinates of the points of the intersection of the curve $y = x^3$ and the straight line $y = 7x - 6$ will give the roots of the equation.

X	**-3**	-2	-1	0	1	2	3
$y=x^3$	**-27**	-8	-1	0	1	8	27

X	**-3**	-2	-1	0	1	2	3
y=7x-6	**-27**	-20	-13	-6	1	8	25

The line $y=7x-6$ intersects the curve in three real points and x- coordinates of the points are 1, 2, -3.

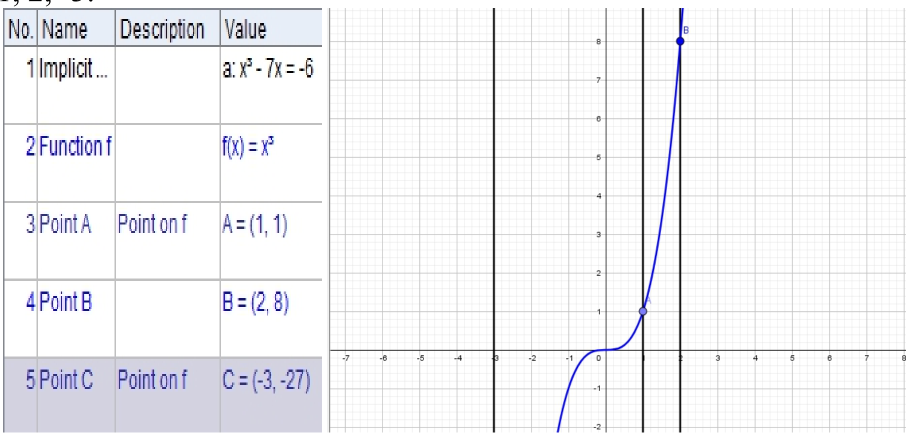


Figure 6 graph of $y = x^3$ and $y = 7x - 6$

Therefore the roots of the equation are 1, 2, -3.

USING CASE 2:

a) $x^3 - 7x + 6 = 0.$

By case 2 ,*multiply the equation by x,we get* $x^4 - 7x^2 + 6x$

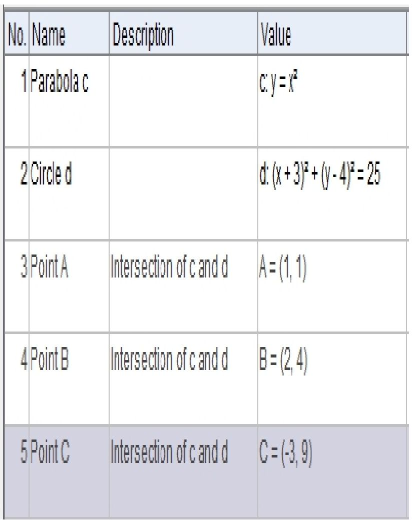
$$\left(x^2\right)^2 + x^2 - 8x^2 + 6x = 0$$

The roots are the x - coordinates of the points of intersection of the curves

$y = x^2$ *and* $y^2 + x^2 - 8y + 6x = 0$

the first curve is a parabola and the second is a circle whose centre is (-3, 4) and the radius 5. By drawing the curves, we can see that the curves intersect at the point whose x- coordinates are 1, 2, -3.

X	**-3**	-2	-1	0	1	2	3
$y=x^2$	**9**	4	1	0	1	4	9



No.	Name	Description	Value
1	Parabola c		c: y = x²
2	Circle d		d: (x + 3)² + (y - 4)² = 25
3	Point A	Intersection of c and d	A = (1, 1)
4	Point B	Intersection of c and d	B = (2, 4)
5	Point C	Intersection of c and d	C = (-3, 9)

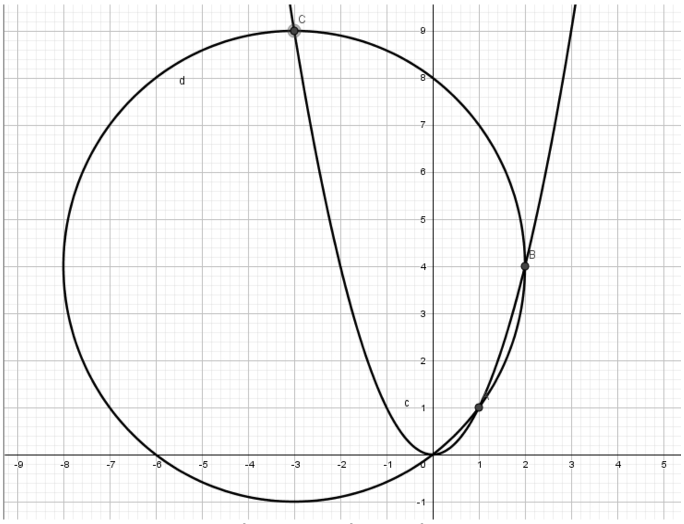


Figure 7 **$y = x^2$** and **$y^2 + x^2 - 8y + 6x = 0$**

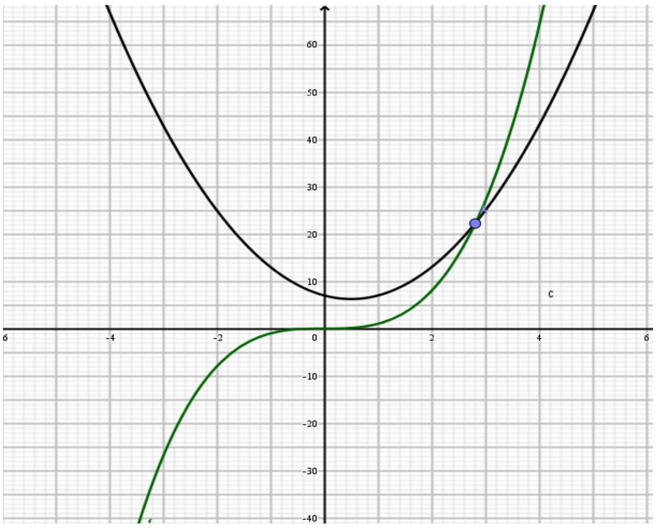
Therefore the roots of the equation are 1, 2 and -3.

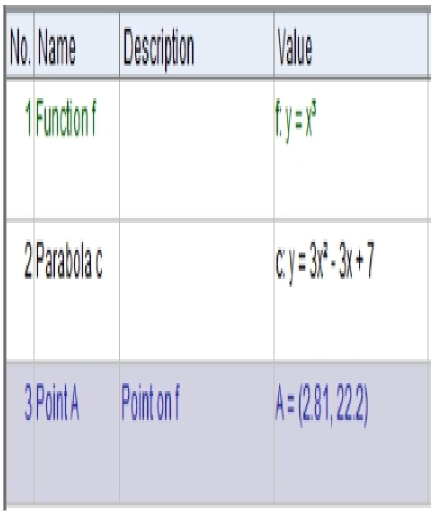
b) Show that the equation $x^3 - 3x^2 + 3x - 7 = 0$ *has any one real root. Find the root graphically to the first decimal place*

By case 1 , the equation can be written as **$x^3 = 3x^2 - 3x + 7$**

Hence the roots are the x- coordinates of the points of intersection of the curves $y = x^3$ and $y = 3x^2 - 3x + 7$.

x	**-3**	-2	-1	0	1	2	3
$y=x^3$	**-27**	-8	-1	0	1	8	27
X	**-3**	-2	-1	0	1	2	3
$y=3x^2-3x+7$	**40**	25	13	0	7	13	25





No.	Name	Description	Value
1	Function f		f: y = x³
2	Parabola c		c: y = 3x² - 3x + 7
3	Point A	Point on f	A = (2.81, 22.2)

Figure 8 graph of $y = x^3$ and $y = 3x^2 - 3x + 7$

If we draw the two curves on a graph paper we will see that the two curves will intersect only in one point and the x- coordinate of that point is 2.8. Hence the equation has only real root and that is 2.8 approximately.

6.3 BIQUADRATIC EQUATIONS (Graphical solution)

Let the equation of the biquadratic be $x^4 + ax^3 + bx^2 + cx + d = 0$
Two conics in general intersect in four points.
Therefore our attempt should be to find two conics, the x- coordinates of whose points of intersection are the roots of the given equation.

The equation can be written as

$$(x^2 + \frac{a}{2}x)^2 - \frac{a^2}{4}x^2 + bx^2 + cx + d = 0$$

$$(x^2 + \frac{a}{2}x)^2 + x^2 + (b - 1 - \frac{a^2}{4})x^2 + cx + d = 0$$

Let $y = x^2 + \frac{a}{2}x$ -------(1)

Then the equation becomes

$$y^2 + x^2 + (b - 1 - \frac{a^2}{4})(y - \frac{a}{2}x) + cx + d = 0$$

$$x^2 + y^2 - \frac{a}{2}(b - 1 - \frac{a^2}{4} - \frac{2c}{a})x + (b - 1 - \frac{a^2}{4})y + d = 0 \quad \text{-------(2)}$$

The equation (1) represents a parabola and (2) a circle.
Trace the curve on a graph paper and the x- coordinates of the points of intersection are the roots of the given equation.

Example 1 :
Solve the biquadratic equation $x^4 - 2x^3 + 4x^2 + 6x - 19 = 0$ *graphical ly*

Solution

The equation can be written as $\left(x^2 - x\right)^2 + 3x^2 + 6x - 19 = 0$

Let $y = x^2 - x$ ------(1)

Then the equation becomes $y^2 + x^2 + 2x^2 + 6x - 19 = 0$

$$x^2 + y^2 + 2\left(y + x\right) + 6x - 19 = 0$$

$$x^2 + y^2 + 8x + 2y - 19 = 0$$

$$\left(x + 4\right)^2 + \left(y + 1\right)^2 = 36$$

$$\left(x + 4\right)^2 + \left(y + 1\right)^2 = 6^2 \quad \text{---------} (2)$$

Trace the curves (1) and (2) on a graph paper.

x	**-3**	-2	-1	0	1	2	3
$y=x^2-x$	**11**	6	2	0	0	2	6

No.	Name	Description	Value	Caption
1	Parabola c		c: $y = x^2 - x$	parabola
2	Circle d		d: $(x + 4)^2 + (y + 1)^2 = 36$	circle
3	Point A	Intersection of c and d	A = (1.64, 1.05)	
4	Point B	Intersection of c and d	B = (-1.69, 4.54)	

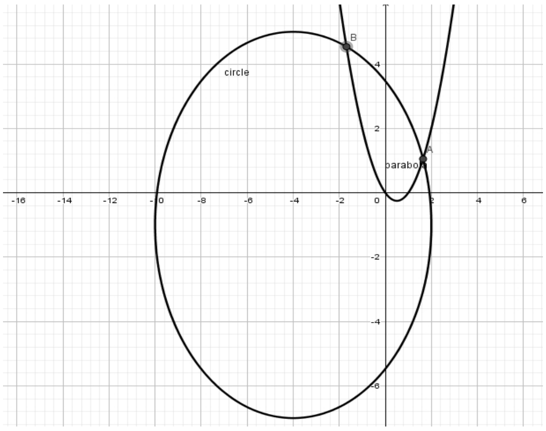


Figure 9 graph of $x^4 - 2x^3 + 4x^2 + 6x - 19 = 0$

The curves intersect only in two real points.
Therefore the given equation has only two real roots and they are approximately 1.6 and -1.7.

6.4 BIQUADRATIC EQUATIONS

Biquadratic equations are quadratic equations with no odd-degree terms:

$$ax^4 + bx^2 + c = 0$$

6.4.1 SOLVING BIQUADRATIC EQUATIONS

To solve biquadratic equations, change; this generates a quadratic equation with the unknown, t : $at^2 + bt + c = 0$

For every positive value of **t** there are **two values of x**, find:

$x = \pm\sqrt{t}$ $\qquad x^4 - 13x^2 + 36 = 0$

$x^2 = t$ $\qquad t^2 - 13t + 36 = 0$

$$t = \frac{13 \pm \sqrt{169 - 144}}{2} = \frac{13 \pm 5}{2}$$

$$t_1 = \frac{18}{2} = 9, t_2 = \frac{8}{2} = 4$$

$$x^2 = 9 \Rightarrow x = \pm 3$$
$$x^2 = 4 \Rightarrow x = \pm 2$$

The same procedure can be used to solve the equation of the type:

$ax^6 + bx^3 + c = 0$

$ax^8 + bx^4 + c = 0$

$ax^{10} + bx^5 + c = 0$

Example 1

Solve $x^6 - 7x^3 + 6 = 0$

Solution

Let $x^3 = t$

$t^2 - 7t + 6 = 0$

$$t = \frac{7 \pm \sqrt{49 - 24}}{2} = \frac{7 \pm 5}{2}$$

$$t_1 = \frac{12}{2} = 6, t_2 = \frac{2}{2} = 1$$

$x^3 = 6 \quad \Rightarrow \quad x = \sqrt[3]{6}$, $\quad x^3 = 1 \quad \Rightarrow \quad x = \sqrt[3]{1} = 1$

Example 2

Solve $x^4 - 10x^2 + 9 = 0$

Solution

Let $x^2 = t$

$t^2 - 10t + 9 = 0$

$$t = \frac{10 \pm \sqrt{10^2 - 36}}{2} = \frac{10 \pm 8}{2}$$

$$t_1 = \frac{18}{2} = 9, t_2 = \frac{2}{2} = 1$$

$x^2 = 9 \quad x = \pm\sqrt{9} = \pm 3$, $x^2 = 1 \quad x = \pm 1$

Example 3

Solve $x^4 - 61x^2 + 900 = 0$

Solution

Let $x^2 = t$

$t^2 - 61t + 900 = 0$

$t = \dfrac{61 \pm \sqrt{3721 - 3600}}{2} = \dfrac{61 \pm 11}{2}$

$t_1 = \dfrac{72}{2} = 36, t_2 = \dfrac{50}{2} = 25$

$x^2 = 36 \quad x = \pm 6$, $x^2 = 25 \quad x = \pm 5$

Example 4

Solve $x^4 - 25x^2 + 144 = 0$

Solution

Let $x^2 = t$

$t^2 - 25t + 144 = 0$

$t = \dfrac{25 \pm \sqrt{625 - 576}}{2} = \dfrac{25 \pm 7}{2}$

$t_1 = \dfrac{32}{2} = 16, t_2 = \dfrac{18}{2} = 9$

$x^2 = 16 \quad x = \pm 4$, $x^2 = 9 \quad x = \pm 3$

Example 5

Solve $x^4 - 16x^2 - 225 = 0$

Solution

Let $x^2 = t$

$t^2 - 16t + 225 = 0$

$t = \dfrac{16 \pm \sqrt{256 + 900}}{2} = \dfrac{16 \pm 34}{2}$

$t_1 = \dfrac{50}{2} = 25, t_2 = \dfrac{-18}{2} = -9$

$x^2 = 25 \quad x = \pm 5$, $x^2 = -9 \quad x = \pm\sqrt{9} \notin R$

6.5 FERRARI'S SOLUTION OF A QUARTIC EQUATION

Example 1.

Factorise $x^4 + 9$.

Solution.

We know that $M^2 - N^2 = (M + N)(M - N)$.

Note that $(x^2 + 3)^2 = x^4 + 6x^2 + 9$, and so

$\left(x^2+3\right)^2-6x^2=x^4+9$

$\left(x^2+3\right)^2-(\sqrt{6}x)^2=x^4+9$

$(\left(x^2+3\right)+\sqrt{6}x)(\left(x^2+3\right)-\sqrt{6}x)=x^4+9$

$(x^2+\sqrt{6}x+3)(x^2-\sqrt{6}x+3)=x^4+9:$

The quadratics can't be factorised any further, and so the full factorisation is

$x^4+9=(x^2+\sqrt{6}x+3)(x^2-\sqrt{6}x+3):$

Factorising a quartic

Let us take the general quartic $x^4+2ax^3+bx^2+2cx+d$. The key to factorising this quartic is to find numbers A, B, and C such that

$x^4+2ax^3+bx^2+2cx+d\equiv\left(x^2+ax+A\right)^2-\left(Bx+C\right)^2$

and then to use the fact that $M^2-N^2=\left(M+N\right)(M-N)$

Example 2

Solve the quadratic equation $x^4+2x^3-x^2-2x-3=0$ *for x*

Solution.

For quadratic, we have $a=1$, and so we must find numbers A, B and C such that

$x^4+2x^3-x^2-2x-3\equiv\left(x^2+x+A\right)^2-\left(Bx+C\right)^2$

Expanding the brackets on the right hand side gives

$\left(x^2+x+A\right)^2=\left(x^2+x+A\right)\left(x^2+x+A\right)$

$(x^2+x+A)^2=(x^2+x+A)(x^2+x+A)$

$=x^4+x^3+Ax^2+x^3+x^2+Ax+Ax^2+Ax+A^2$

$=x^4+2x^3+\left(2A+1\right)x^2+2Ax+A^2$

and $\left(Bx+C\right)^2=B^2x^2+2BCx+C^2$. Now

$\left(x^2+x+A\right)^2-\left(Bx+C\right)^2=x^4+2x^3+\left(2A+1\right)x^2+2Ax+A^2-\left(B^2x^2+2BCx+C^2\right)$

$(x^2+x+A)^2-(Bx+C)^2=x^4+2x^3+(2A+1)x^2+2Ax+A^2-(B^2x^2+2BCx+C^2)$

$=x^4+2x^3+(2A+1-B^2)x^2+2(A-BC)x+A^2-C^2$

Therefore, we must find numbers A, B, and C such that

$x^4+2x^3-x^2-2x-3\quad x^4+2x^3+(2A+1-B^2)x^2+2(A-BC)x+A^2-C^2:$

By comparing coefficients, we see that

$$\begin{cases}-1=2A+1-B^2\Leftrightarrow B^2=2A+2 & (1)\\ -1=A-BC\Leftrightarrow BC=A+1 & (2)\\ -3=A^2-C^2\Leftrightarrow C^2=A^2+3 & (3)\end{cases}$$

Now (1) X (3) gives

$B^2C^2 = (2A+2)(A^2+3) = 2(A+1)(A^2+3)$ and $(2)^2$ gives

$(BC)^2 = (A+1)^2$. We can now eliminate $(BC)^2$ to obtain

$$(A+1)^2 = 2(A+1)(A^2+3)$$

$$2(A+1)(A^2+3)-(A+1)^2 = 0$$

$$(A+1)(2(A^2+3)-(A+1)) = 0$$

$$(A+1)(2A^2+6-A-1) = 0$$

$$(A+1)(2A^2-A+5) = 0:$$

We have a cubic equation for A. We only need to find one solution, so choose A = -1. Now (1) gives $B^2 = 0 \Rightarrow B = 0$ and (3) gives $C^2 = (-1)^2 + 3 = 4 \Rightarrow C = 2$.
Choose one solution, we choose $C = 2$.

Now $x^4 + 2x^3 - x^2 - 2x - 3 \equiv (x^2+x+A)^2 - (Bx+C)^2$

$$= (x^2+x-1)^2 - (0x+2)^2$$

$$= (x^2+x-1+2)(x^2+x-1-2)$$

$$= (x^2+x+1)(x^2+x-3)$$

Hence

$$x^4 + 2x^3 - x^2 - 2x - 3 = 0$$

$$(x^2+x+1)(x^2+x-3) = 0$$

$$\Leftrightarrow (x^2 + x + 1)(x^2 + x - 3) = 0$$

$$\Leftrightarrow (x^2+x+1) = 0 \ or \ (x^2+x-3) = 0$$

$$\Leftrightarrow (x^2+x-3) = 0$$

$$x = \frac{-1 \pm \sqrt{1-4(1)(-3)}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$

The general case

A general quadratic equation $x^4 + 2ax^3 + bx^2 + 2cx + d$, we need to find numbers A, B, and C such that

$$x^4 + 2ax^3 + bx^2 + 2cx + d \equiv (x^2+ax+A)^2 - (Bx+C)^2:$$

When we expand the brackets on the right-hand side, we get

$$x^4 + 2ax^3 + bx^2 + 2cx + d = x^4 + 2ax^3 + (2A+a^2-B^2)x^2 + 2(A-BC)x + A^2 - C^2:$$

By comparing coefficients, we see that

$$\begin{cases} \text{b} = 2A + \text{a}^2 - B^2 \Leftrightarrow B^2 = 2A + \text{a}^2 - b \ (1) \\ \text{c} = A - BC \Leftrightarrow BC = A - c \ (2) \\ \text{d} = A^2 - C^2 \Leftrightarrow C^2 = A^2 \text{ - d } (3) \end{cases}$$

Now (1) X (3) $\Rightarrow$ $B^2C^2 = (2A+a^2-b)(A^2-d)$ and $(2)^2$ gives $(BC)^2 = (A-c)^2$. We can now eliminate $(BC)^2$ to obtain $(A-c)^2 = (2A+a^2-b)(A^2-d)$

The cubic equation in terms of A, find the 3 solutions using the following

$\left(x^2+ax+A\right)^2-\left(Bx+C\right)^2 = \left[x^2+ax+A+ \ (Bx+C\ )\right][x^2+ax+A \ - (Bx+C\ )]$

we can factorize the given equation by factorizing the two quadratics.

6.6 SOLUTION OF BIQUADRATIC (SIMPLE PROCEEDURE)

Let the equation be $x^4 + \ 2ax^3 + bx^2 + \ 2cx + d = \ 0$

Express the left side of the equation as the difference of squares of a quadratic function and a linear function.

The equation can be written as

$$\left(x^2+\frac{a}{2}x\right)^2+(b-\frac{a^2}{4})x^2+cx+d=0$$

This equation can be expressed as

$$\left(x^2+\frac{a}{2}x+\lambda\right)^2+\{(\frac{a^2}{4}-b+2\lambda)x^2+(\lambda a-c)x+\lambda^2-d\}=0$$

Which of the form

$$\left(x^2+\frac{a}{2}x+\lambda\right)^2+(\alpha x-\beta)^2=0$$

Where $\alpha^2=\frac{a^2}{4}-b+2\lambda, 2\alpha\beta=\lambda a-c, \beta^2=\lambda^2-d$

Eliminating α and β from these equation in λ is real. From the real root of this equation α and β can be determined.

Hence the given equation can be factorised into

$$x^2+\frac{a}{2}x+\lambda\pm(\alpha x+\beta)=0$$

Solving the two quadratic equations, all the four roots of the biquadratic can be determined. Hence the solution of the biquadratic equation depends on the solution of a cubic equation, which can be solved by Cardon's method.

Example 1

Solve the equation $4x^4 + \ 4x^3 - 7x^2 - 4x - 12 = \ 0$

Solutions

This equation can be written as $(2x^2 + x)^2 - 8x^2 - 4x - 12 = 0$

$(2x^2 + x + \lambda)^2 - \{(4\lambda + 8)x^2 + (2\lambda + 4)x + \lambda^2 + 12\} = 0$

$(2x^2 + x + \lambda)^2 - (\alpha x + \beta)^2 = 0$

Where $\alpha^2 = 4(\lambda + 2)$, $2\alpha\beta = 2(\lambda + 2)$, $\beta^2 = \lambda^2 + 12$

Eliminating α and β from these relations, we get

$16(\lambda+2)(\lambda^2+12) = 4(\lambda+2)^2$

Which reduces to

$(\lambda+2)(4\lambda^2 - \lambda + 46) = 0$

The only real root of the equation is -2.

Hence $\alpha = 0$, $\beta = \pm 4$.

Hence the given equation reduces to

$(2x^2 + x - 2)^2 - (4)^2 = 0$

$(2x^2 + x + 2)(2x^2 + x - 6) = 0$

The roots of $(2x^2 + x - 2) = 0$ are $\dfrac{-1 \pm i\sqrt{15}}{4}$

The roots of $(2x^2 + x - 2) = 0$ are -2 and $\dfrac{3}{2}$.

The roots of the given biquadratic equation are

-2, $\dfrac{3}{2}$ and $\dfrac{-1 \pm i\sqrt{15}}{4}$.

Example 2

Solve the equation $x^4 - 4x^3 - 10x^2 + 64x + 40 = 0$

Solution

The equation can be written as

$(x^2 - 2x)^2 - 14x^2 + 64x + 40 = 0$

and hence as

$(x^2 - 2x + \lambda)^2 - \{(2\lambda + 14)x^2 - (4\lambda + 64)x + \lambda^2 - 40\} = 0$

i.e., $(x^2 - 2x + \lambda)^2 - (\alpha x + \beta)^2 = 0$

where $\alpha^2 = 2(\lambda + 7)$, $2\alpha\beta = -4(\lambda + 16)$, $\beta^2 = \lambda^2 - 40$.

Eliminating α and β from these relations, we get

$8(\lambda + 7)(\lambda^2 - 40) = 16(\lambda+16)^2$

On simplification, this equation reduces to

$\lambda^3 + 5\lambda^2 - 104\lambda - 792 = 0$

$792 = 2^3 3^2 .11$.

On trial we find that 11 satisfies the equation. Dividing by $\lambda - 11$, we get

$\lambda^2 + 16\lambda + 72 = 0$ which gives imaginary roots.

When $\lambda = 11$, $\alpha = \pm 6$, $\beta = \pm 9$, $\alpha\beta = -54$

$\alpha = 6, \beta = -9$ or $\alpha = -6, \beta = 9$

Both pairs of values will lead to the same factorization of the expression on the left side.

Hence the equation reduces to

$(x^2 - 2x + 11)^2 - (6x-9)^2 = 0$

$(x^2 + 4x + 2)(x^2 - 8x + 20) = 0$

Hence the root of the given equation are -2 $\pm\sqrt{2}$ and 4 $\pm$2i.

Example 3

Solve the equation $2x^4 + 6x^3 - 3x^2 + 2 = 0$.

Solution
Transform this equation into another whose roots are twice the roots of the given equation.
The transformed equation is $2x^4 + 6(2)x^3 - 3(2^2)x^2 + 2(2^4) = 0$.
Which reduces to $x^4 + 6x^3 - 6x^2 + 16 = 0 ----(1)$
This equation can be written as $(x^2 + 3x + \lambda)^2 - \{(2\lambda + 15)x^2 + 6\lambda x + \lambda^2 - 16\} = 0$
This equation can be expressed as
$(x^2 + 3x + \lambda)^2 - (\alpha x + \beta)^2 = 0$ where $\alpha^2 = 2\lambda + 15, 2\alpha\beta = 6\lambda, \beta^2 = \lambda^2 - 16$
Eliminating α and β from these equations, we get
$4(2\lambda + 15)(\lambda^2 - 16) = 36\lambda^2$
Simplifying we get $2\lambda^3 + 6\lambda^2 - 32\lambda - 240 = 0 ----(2)$
240 = 2^4(3)(5), by trial we see that $\lambda = 5$ is a root of (2)
Hence $\alpha = \pm 5$, $\beta = \pm 3, \alpha\beta = 15$
Therefore $\alpha = 5$, $\beta = 3$ or $\alpha = -5$, $\beta = -3$.
Hence for both pairs of values (1) reduces to
$(x^2 + 3x + 5)^2 - (5x + 3)^2 = 0$
$(x^2 + 8x + 8)(x^2 - 2x + 2) = 0$
The roots of this equation are -4 $\pm 2\sqrt{2}$ and 1 $\pm$ i. hence the roots of the given equation are -2 $\pm\sqrt{2}$ and $\frac{1 \pm i}{2}$.

PROBLEMS FOR PRACTICE

Solve the biquadratic equations

1. $x^4 + 12x - 5 = 0$ *answer* $-1 \pm \sqrt{2}, 1 \pm 2i$
2. $x^4 + 4x^3 - x^2 + 8x + 4 = 0$ *answer* $\frac{-5 \pm \sqrt{13}}{2}, \frac{1 \pm i\sqrt{7}}{2}$
3. $3x^4 - 10x^3 + 6x^2 - 10x + 3 = 0$ *answer* $\pm i$, 3, 1/3.
4. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ *answer* 1, 2, 3, 4
5. $x^4 - 18x^2 + 32x - 15 = 0$ *answer* 1,1, 3, -5
6. $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$ *answer* 3 ,3, -1
7. $x^4 + 2x^3 + 14x + 15 = 0$ *answer* -1 , $-3, 1 \pm 2i$

6.6 SOLVING CUBIC EQUATIONS

Now let us move on to the solution of cubic equations. Like a quadratic, a cubic should always be re-arranged in to its standard form, in this case

$ax^3 + bx^2 + cx + d = 0$

The equation $x^2 + 4x - 1 = \frac{6}{x}$

is a cubic, though it is not written in the standard form. We need to multiply through by x, giving us $x^3 + 4x^2 - x = 6$
and then we subtract 6 from both sides, giving us

$x^3 + 4x^2 - x - 6 = 0$

This is now in the standard form
When solving cubics it helps if you know one root to start with.

We have to recall the following , before finding the solution of the equations

6.6.1 THE REMAINDER THEOREM:

- When we divide a polynomial $f(x)$ by $x - c$ the remainder equals f(c)

6.6.2 THE FACTOR THEOREM:

- When $f(c) = 0$ then $x - c$ is a factor of the polynomial
- When $x - c$ is a factor of the polynomial then $f(c) = 0$

Example 1

Solve $x^3 - 5x^2 - 2x + 24 = 0$, *given that* $x = -2$ *is a solution.*

Solution

By **Factor Theorem,** It states that if $x = -2$ is a solution of this equation, then $x + 2$ is a factor of this whole expression.

This means that $x^3 - 5x^2 - 2x + 24 = 0$ can be written in the form

$$\left(x + 2\right)\left(x^2 + ax + b\right) = 0 \quad \text{where } a \text{ and } b \text{ are numbers.}$$

To find a and b, and we do this by a process called **synthetic division**.

$$\begin{array}{rrrr|l} 1 & -5 & -2 & 24 & x = -2 \\ & -2 & 14 & -24 & \\ \hline 1 & -7 & 12 & 0 & \end{array}$$

Therefore $x = -2$ is a root of the given equation.

$x^2 - 7x + 12 = 0$

$\left(x + 2\right)(x - 3)(x - 4) = 0$

The solutions are $x = -2, 3$ or 4.

Example 2

Solve $x^3 - 7x - 6 = 0$, *given is* $x = 1$.

Solution
Substituting $x = 1$ in the left-hand side we find
$1^3 - 7(1) - 6 = -12$

$x = 1$ is clearly not a solution.
we try the value $x = -1$: $(-1)^3 - 7(-1) - 6 = 0$, so $x = -1$ is a solution. Therefore $x + 1$ is a factor.
The cubic can be written in the form $(x + 1)(x^2 + ax + b) = 0$
We can perform **synthetic division** to find the other factors.

$$\begin{array}{rrrr|l} 1 & 0 & -7 & -6 & x = -1 \\ & -1 & 1 & 6 & \\ \hline 1 & -1 & -6 & 0 & \end{array}$$

The quadratic is $x^2 - x - 6$
$(x - 3)(x + 2) = 0$
The solutions are $x = -2, -1$ or 3.

Example 3

Solve the equation $x^3 - 4x^2 - 9x + 36 = 0$.

Solution
The cubic can be factorised as $x^3 - 4x^2 = x^2(x - 4)$.
$x^3 - 4x^2 - 9x + 36 = 0$
$x^2(x - 4) - 9(x - 4) = 0$
$(x^2 - 9)(x - 4) = 0$
$(x + 3)(x - 3)(x - 4) = 0$ ------------ (1)
The solutions are $x = -3$, 3 or 4

Example 4

Solve the equation $x^3 - 6x^2 - 6x - 7 = 0$

Solution

If there is to be a solution, then because the coefficient of x^3 is 1, it is going to be an integer and it's going to be a factor of 7. This leaves us with only four possibilities: $1, -1, 7,$ and -7.

$$\begin{array}{rrrr|l} 1 & -6 & -6 & -7 & x = 7 \\ & 7 & 7 & 7 & \\ \hline 1 & 1 & 1 & 0 & \end{array}$$

$x = 7$ is a root. The equation is $x^2 + x + 1 = 0$ has no real solutions, so the only possible solution to this cubic is $x = 7$.

Example 5

Solve the equation $x^3 + 3x^2 + 3x + 1 = 0.$

Clue: This has the widely-known factorization $(x + 1)^3 = 0$ from which we have the root $x = -1$ repeated three times.

PROBLEMS FOR PRACTICE

1. *For each of the following cubic equations one root is given. Determine the other roots of each cubic.*

$a) x^3 + 3x^2 - 6x - 8 = 0$ *has a root at* $x = 2.$

$b) x^3 + 2x^2 - 21x + 18 = 0$ *has a root at* $x = 3.$

$c) x^3 + 4x^2 + 7x + 6 = 0$ *has a root at* $x = -2.$

$d) 2x^3 + 9x^2 + 3x - 4 = 0$ *has a root at* $x = -4.$

2. *For each of the following cubic equations use synthetic division to* determine *if the given value of x is a root of the equation. Where it is,* determine *the other roots of the equation.*

$a)$ *Is* $x = -2$ *a root of the equation* $x^3 + 9x^2 + 26x + 24 = 0$?

$b)$ *Is* $x = 4$ *a root of the equation* $x^3 - 6x^2 + 9x + 1 = 0$?

$c)$ *Is* $x = -1$ *a root of the equation* $x^3 + 6x^2 + 3x - 5 = 0$?

$d)$ *Is* $x = 2$ *a root of the equation* $x^3 + 2x^2 - 20x + 24 = 0$?

6.7 GENERAL SOLUTION OF THE CUBIC EQUATIONS : CARDON'S METHOD

Cardon's method provides a technique for solving the general cubic equation $ax^3 + bx^2 + cx + d = 0$ in terms of radicals. As with the quadratic equation, it involves a "discriminant" whose sign determines the number (1, 2, or 3) of real solutions. However, its implementation requires substantially more technique than does the quadratic formula. For example, in the "irreducible case" of three real solutions, it calls for the evaluation of the cube roots of complex numbers.

In outline, Cardon's methods involves the following steps:

1. "Eliminate the square term" by the substitution y = x + b/3a. Rather than keeping track of such a substitution relative to the original cubic, the method often begins with an equation in the reduced form

$$x^3 + px + q = 0. \text{---------} (1)$$

2. Letting x = u+v,

$$(u+v)^3 + p(u+v) + q = 0$$

$\mathbf{u^3 + v^3 + 3uv(u+v) + p(u + v) + q = 0}$

rewrite the above equation as

$\mathbf{u^3 + v^3 + (u+v)(3uv + p) + q = 0}$.

3. Setting 3uv + p = 0, the above equation becomes $u^3 + v^3 = -q$. In this way, we obtain the system

$\mathbf{u^3 + v^3 = -q}$ **--------(2)**

$\mathbf{3uv + p = 0}$ **---------(3)**

4. Eliminate u from (2) and (3). We get

u = -p/3v ---------(4)

$(\frac{-p}{3v})^3 + v^3 + q = 0$ -------(5)

$-\frac{p^3}{27} + v^6 + v^3 q = 0$

$v^6 + qv^3 - \frac{p^3}{27} = 0$ ---------(6)

5. Similarly, eliminate v from (2) and (3), we get

$u^6 + qu^3 - \frac{p^3}{27} = 0$ ---------(7)

6. From (6) and (7) relations, we get that u^3 and v^3 are the roots of the equation

$t^2 + qt - \frac{p^3}{27} = 0$

7. u^3 and v^3 can be determined from this equation

$u^3 v^3 = -\frac{p^3}{27} \quad \Rightarrow \quad uv = -\frac{p}{3}$

8. In order to find u and v, we are now obligated to find the cube roots of these solutions.

(i) In the case $\mathbf{27q^2 + 4p^3 < 0}$

this entails finding the cube roots of complex numbers.

(ii) Even in the case $27q^2 + 4p^3 > 0$, there are some unexpected wrinkles.

These are illustrated by the equation **$x^3 + x^2 - 2 = 0$**

for which x = 1 is clearly a solution. Although Cardon's method enables one to find this root without confronting cube roots of complex numbers, it displays the solution x = 1 in the rather obscure form

It is against this historical background that an iteration-based alternative to Cardon's method at the pre-calculus level, one that is derived from "a Babylonian technique for finding cube roots."

6.7.1 Examples Using Cardon's Method to Solve Cubic Equations

1. $64x^3 - 48x^2 + 12x - 1$
2. $x^3 + 6x^2 + 11x + 6$
3. $x^3 + 3x^2 + 3x - 2$
4. $x^3 + 2x^2 + 10x - 20$
5. x巢x $+10x$ �
6. $x^3 + 4x^2 + x - 6$
7. $1000x^3 - 1254x^2 - 496x + 191$

1. $64x^3 - 48x^2 + 12x - 1$

Solution

Cubic in normal form: $x^3 - 0.75x^2 + 0.1875x - 0.015625$

$x^3 + ax^2 + bx + c$ ----[1]

Substitute $x = t - \frac{a}{3}$, to eliminate the x^2 term

New equation is $t^3 + pt + q = 0$

Where $p = b - \frac{a^2}{3}$, and $q = c + \frac{(2a^3 - 9ab)}{27}$

$p = 0$, $q = 0$

Therefore the equation becomes: $t^3 + 0t + 0$----[2]

As $p \ and \ q$ are zero, then **all the roots are real and equal** to $-\frac{a}{3(0.25)}$

The roots for given equation are: $x_1 = 0.25$, $x_2 = 0.25$, $x_3 = 0.25$

2. $x^3 + 6x^2 + 11x + 6$

Solution

Cubic in normal form: $x^3 + 6x^2 + 11x + 6$

$x^3 + ax^2 + bx + c$ $----[1]$

Substitute $x = t - \frac{a}{3}$, to eliminate the x^2 term

New equation is $t^3 + pt + q = 0$

Where $p = b - \frac{a^2}{3}$, and $q = c + \frac{(2a^3 - 9ab)}{27}$

$p = -1$, $q = 0$

Therefore the equation becomes: $t^3 - 1t + 0 ----[2]$

As $p \ and \ q$ are zero, then **all the roots are real and equal** to

$x_1 = -\frac{a}{3}, x_2 = -\frac{a}{3 \pm \sqrt{(-p)}}$

As p < 0, then all the roots are real

The roots for given equation are: $x_1 = -2, x_2 = -1, x_3 = -3$

3. $x^3 + 3x^2 + 3x - 2$

Solution

Let $x^3 + ax^2 + bx + c$ $----[1]$

Substitute $x = t - \frac{a}{3}$, to eliminate the x^2 term

New equation is $t^3 + pt + q = 0$

Where $p = b - \frac{a^2}{3}$, and $q = c + \frac{(2a^3 - 9ab)}{27}$

$p = 0$, $q = -3$

Therefore the equation becomes: $t^3 + 0t - 3 ----[2]$

$t = \sqrt[3]{-q} = \sqrt[3]{3}$

$x = t - \frac{a}{3} = \sqrt[3]{3} - \frac{a}{3} = \sqrt[3]{3} - 1$

$$x_1 = 0.4422495703 ,$$

The roots for given equation are: $x_2 = -1.7211247852 + 1.2490247665i$,

$$x_3 = -1.7211247852 - 1.2490247665i$$

4. $x^3 + 2x^2 + 10x - 20$

In 1225, Leonardo da Pisa found the root 1.368808107 of this equation. For his time it is intriguing, especially because this a truncated value. Rounded to 9 places, it is 1.36880810**8**.

Let $x^3 + ax^2 + bx + c$ $----[1]$

Substitute $x = t - \frac{a}{3}$, to eliminate the x^2 term

New equation is $t^3 + pt + q = 0$

Where $p = b - \frac{a^2}{3}$, and $q = c + \frac{(2a^3 - 9ab)}{27}$

p = 8.666666666666 , q = -26.07407407407406

Therefore the equation becomes: $t^3 + 8.66666666666t - 26.07407407407406 ----[2]$

$The\ discri\min ant\ of\ the\ cubic,\ D = (p/3)^3 + (q/2)^2$

$Before\ rounding,\ D = 194.0740740740741$

$D = 194.0740740740741$

$D > 0 \rightarrow 2\ complex\ roots$

$D = 0 \rightarrow real\ roots,\ with\ two\ the\ same$

$\mathrm{D} < 0 \rightarrow$ Three real roots

Because $\mathrm{D} > 0$, two of the roots are compl ex

Our reduced equation is :

$\mathrm{t}^3 + 8.666666666666666\mathrm{t} - 26.074074074074076$[**2**, **repeated**]

Or in algebra, $\mathrm{t}^3 + \mathrm{pt} + \mathrm{q} = 0$

There is a relationship :

$(\mathrm{u} - \mathrm{v})^3 + 3\mathrm{uv}(\mathrm{u} - \mathrm{v}) - (\mathrm{u}^3 - \mathrm{v}^3) = 0,$

which indicates how we might solve the e quation

If we set $\mathrm{t} = \mathrm{u} - \mathrm{v}$, $\mathrm{p} = 3\mathrm{uv}$, or $\mathrm{v} = \mathrm{p}/3\mathrm{u}$ and $\mathrm{q} = -(\mathrm{u}^3 - \mathrm{v}^3)$, that is :

$\mathrm{v} = 8.666666666666666 / (3\mathrm{u}) ---------[$ **4**]

$\mathrm{u}^3 - (8.666666666666666 / (3*\mathrm{u}) - 26.0740740\,74074076 = 0\ ---------[$**5**]

So $\mathrm{u}^3 - \mathrm{p}^3 / (27\mathrm{u}^3) + \mathrm{q} = 0$

Hence, We need to find numbers, u and v, such that :

$\mathrm{u}^3 - \mathrm{v}^3 = -\mathrm{q}$, and $\mathrm{uv} = \mathrm{p}/3$, giving $\mathrm{t} = \mathrm{v} - \mathrm{u}$

$\mathrm{u}^3 - \mathrm{v}^3 = 26.074074074074076 -----------[$ **6**]and

$\mathrm{uv} = 8.666666666666666/3\ -------------[7]$

Using $\mathrm{v} = 8.666666666666666/3\mathrm{u}$, we can el iminate u

$\mathrm{u}^3 - \mathrm{p}^3/27\mathrm{u}^3 = -\mathrm{q}$

The resulting equation is :

$27\mathrm{u}^6 + 27\mathrm{qu}^3 - \mathrm{p}^3 = 0$

Normalised, this is $\mathrm{u}^6 + \mathrm{qu}^3 - \mathrm{p}^3/27 = 0$

$\mathrm{u}^6 + 26.074074074074076 * \mathrm{u}^3 - 8.666666\,666666666^3/27 = 0$

This is a quadratic equation in u^3, so we know how to solve it

$\mathrm{u}^3 = -\mathrm{q}/2 \pm \sqrt{}(\mathrm{q}^2/4 + \mathrm{p}^3/27)\ (5)$

The square root may be negative, making the u's complex numbers, although the re sulting roots might not be complex.

$\mathrm{D} = 679.857338820302/4 + 650.9629629629628\ /27 = 194.0740740740741$

After rounding, $\mathrm{D} = 194.0740740740741$

The square root above isn't negative, so two of the roots are complex

$u = \sqrt[3]{\left(-q/2+\sqrt{(D)}\right)} = 2.9988174659594593$

$v = \sqrt[3]{\left(q/2+\sqrt{(D)}\right)} = 0.9633426914714203$

$x_1 = u_1 - v_1 - \frac{a}{3} = 1.3688081078213723, 0$

$x_2 = -0.5*(u_1 - v_1) - a/3 + (u_1 + v_1) * \sqrt{(3)}/2i$

$x_2 = -1.6844040539 + 3.4313313502i$

Now we know x_2, we also know that x_3 simply has the opposite sign imaginary part:

So $x_3 = \text{Re}(x_2) - \text{Im}(x_2) = -1.684404053910686 - 3.4313313501976\ 923i$, but here is the formula: $x_3 = -0.5 * (u1 - v1) - a/3 + (u1 + v1) * \sqrt{(3)}/2i$

$x_3 = -1.6844040539 - 3.4313313502i$

The roots are

$x_1 = 1.3688081078$

$x_2 = -1.6844040539 + 3.4313313502i$

$x_3 = -1.6844040539 - 3.4313313502i$

5. x巢x $+10x$ �

Solution

Cubic in normal form: x巢x $+10x$ �

Let $x^3 + ax^2 + bx + c$ ----[1]

Substitute $x = t - \frac{a}{3}$, to eliminate the x^2 term

New equation is $t^3 + pt + q = 0$

Where $p = b - \frac{a^2}{3}$, and $q = c + \frac{(2a^3 - 9ab)}{27}$

$p = 9.666666666666$, $q = -6.259259259259259$

Therefore the equation becomes: $t^3 + 9.66666666666t - 6.259259259259259$ ----[2]

The discriminant of the cubic, $D = (p/3)^3 + (q/2)^2$

Before rounding, $D = 43.24999999999986$

$D > 0 \rightarrow$ *2 complex roots*

$D = 0 \rightarrow$ *real roots, with two the same*

$D < 0 \rightarrow$ Three real roots

Because $D > 0$, two of the roots are compl ex

Our reduced equation is :

$t^3 + 9.666666666666666t - 6.259259259259259$

Or in algebra, $t^3 + pt + q = 0$

There is a relationship:

$(u - v)^3 + 3uv(u - v) - (u^3 - v^3) = 0$, which indicates how we might solve the equation

If we set $t = u - v$, $p = 3uv$, or $v = p/3u$ and $q = -(u^3 - v^3)$, that is:

$v = 9.666666666666666/(3u)$ --------------**[4]**

$u^3 - (9.666666666666666/(3*u) - 6.2592592592592595 = 0$ ---------- **[5]**

So $u^3 - p^3/(27u^3) + q = 0$

Hence, We need to find numbers, u and v, such that:

$u^3 - v^3 = -q$, and $uv = p/3$, giving $t = v - u$

$u^3 - v^3 = 6.2592592592592595$-----------------**[6]** and

$uv = 9.666666666666666/3$ -------------**[7]**

Using $v = 9.666666666666666/3u$, we can eliminate u

$u^3 - p^3/27u^3 = -q$

The resulting equation is:

$27u^6 + 27qu^3 - p^3 = 0$

Normalized, this is

$u^6 + qu^3 - p^3/27 = 0$

$= u^6 + 6.2592592592592595 * u^3 - 9.666666666666666^3/27 = 0$

This is a quadratic equation in u^3, so we know how to solve it

$u^3 = -q/2 \pm \sqrt{(q^2/4 + p^3/27)}$ ---------------- (8)

The square root may be negative, making the u 's complex numbers, although the resulting roots might not be complex.

$D = 39.17832647462277/4 + 903.2962962962962/27 = 43.24999999999999$

After rounding,

$D = 43.24999999999999$

The square root above isn't negative, so two of the roots are complex

$u = \sqrt[3]{\left(-q/2 + \sqrt{(D)}\right)} = 2.133118405301851,$

$v = \sqrt[3]{\left(q/2 + \sqrt{(D)}\right)} = 1.510568852724448$

$x1 = u1 - v1 - a/3 = 0.28921621924406943,0$

$x2 = -0.5*(u1 - v1) - a/3 + (u1 + v1)*\sqrt{(3)}/2i$

$x2 = -0.6446081096 + 3.1555257289i$

Now we know x2, we also know that x3 simply has the opposite sign imaginary part:

So $x3 = Re(x2) - Im(x2) = -0.6446081096220346 - 3.1555257288964396i$,

but here is the formula:

$x3 = -0.5 * (u1 - v1) - a/3 + (u1 + v1) * \sqrt{(3)}/2i$

$x3 = -0.6446081096 - 3.1555257289i$

In summary, the roots for

Cubic: $x^3 + x^2 + 10x - 3$ are:

$x1=0.2892162192$ (Remainder=0)
$x2= -0.6446081096+3.1555257289i$ (Remainder=0)
$x3= -0.6446081096-3.1555257289i$ (Remainder=0)
The remainder is the result of substituting the value in the equation, rounded to 10 decimal places

6. $x^3 + 4x^2 + x - 6$

Cubic in normal form: $x^3 + 4x^2 + x - 6$
$x^3 + ax^2 + bx + c$ ---------- **[1]**
Substitute $x = t - a/3$, to eliminate the x^2 term
New equation $t^3 + pt + q = 0$
Where $p = b - a^2/3$, and $q = c + (2a^3 - 9ab) / 27$
$p = -4.333333333333333$, $q = -2.5925925925925926$
So the new equation is:
$t^3 - 4.333333333333333t - 2.5925925925925926$ --------- **[2]**
The discriminate of the cubic, $D = (p/3)^3 + (q/2)^2$
Before rounding, $D = -1.3333333333333335$
$D = -1.333333333333333$
$D > 0 \rightarrow$ 2 complex roots
$D = 0 \rightarrow$ real roots, with two the same
$D < 0 \rightarrow$ Three real roots
Because $D < 0$, all the roots are real and distinct
Our reduced equation is:
$t^3 - 4.333333333333333t - 2.5925925925925926$ **[2, repeated]**
Or in algebra, $t^3 + pt + q = 0$
There is a relationship:
$(u - v)^3 + 3uv(u-v)-(u^3-v^3)=0$, which indicates how we might solve the equation
If we set $t = u-v$, $p = 3uv$, or $v = p/3u$ and $q = -(u^3 - v^3)$, that is:
$v = -4.333333333333333/(3u)$ -----------**[4]**
$u^3 - (-4.333333333333333/(3*u) - 2.5925925925925926 = 0$ ----------**[5]**
So $u^3 - p^3/(27u^3) + q = 0$
Hence, We need to find numbers, u and v, such that:
$u^3 - v^3 = -q$, and $uv = p/3$, giving $t = v - u$
$u^3 - v^3 = 2.5925925925925926$--------------- **[6]** and
$uv = -4.333333333333333/3$ -------------------**[7]**
Using $v = -4.333333333333333/3u$, we can eliminate u
$u^3 - p^3/27u^3 = -q$
The resulting equation is:
$27u^6 + 27qu^3 - p^3 = 0$
Normalised, this is
$u^6 + qu^3 - p^3/27 = 0$
$= u^6 + 2.5925925925925926 * u^3 + 4.333333333333333^3/27 = 0$
This is a quadratic equation in u^3, so we know how to solve it
$u^3 = -q/2 \pm \sqrt{(q^2/4 + p^3 / 27)}$ (5)
The square root may be negative, making the u's complex numbers, although the resulting roots might not be complex.
$D = 6.72153635116598 / 4 - 81.37037037037035/27 = -1.3333333333333326$
After rounding, $D = -1.333333333333332$

The square root is a complex number, because $D < 0$
We need to find the cube roots of $u^3 = 1.2962962963 + 1.1547005384i$
Convert to trigonometric form
$D = (q/2)^2 + (p/3)^3$
So $u^3 = -q/2 + \sqrt{(-D)}\, i$
As $u^3 - v^3 = -q$,
$v^3 = q/2 + \sqrt{(-D)}\, i$
r is therefore the same for both u and v [A]
$r^2 = (-q/2)^2 - D$
$= (-q/2)^2 - (q/2)^2 + (p/3)^3)$
$= -(p/3)^3)$
$r = \sqrt{(-(p/3)^3)}$
Use cos to find φ, as this makes finding the angles easier (with a calculator or computer)
$r = \sqrt{((4.333333333333333/3)^3)}$ (as it is a phasor, it can have positive values only.)
$r = 1.7360061696678466$ and $r^{1/3} = 1.2018504251546631$
$\cos(\varphi) = -q/2r$
$\cos(\varphi) = 2.5925925925925926/(2*1.7360061696678466)$
For v, $\cos(\varphi_v) = -2.5925925925925926/(2*1.7360061696678466)$
So, $\cos(\varphi_v) = -\cos(\varphi)$ [B]
Similarly, $\sin(\varphi_v) = \sin(\varphi)$ [C]
$\operatorname{acos}(\varphi) = -q/(2*r)$
$\varphi = 0.7276916222864558$
$x_1 = u - v - a/3$
$u = r^{1/3} * (\cos(\varphi/3) + i\sin(\varphi/3)$
$v = r^{1/3} * (-\cos(\varphi/3) + i\sin(\varphi/3)$ (from equations B and C above)
So, $x_1 = 2*r*\cos(\varphi) - a/3$ (because the sines cancel and the cosines are doubled
$x_1 = 2*1.2018504251546631*\cos(0.7276916222864558) - 4/3$
We find the other x's by adding 2π to φ, and dividing the result by 3
$\varphi_2 = \varphi + 2*\pi$
$x2 = 2 * 1.2018504251546631 * \cos(2.3369589764886807) - 4/3$
$\varphi_2 = \varphi + 4 * \pi$
$x_3 = 2 * 1.2018504251546631 * \cos(4.431354078881876) - 4/3$
In summary, the roots for Cubic: $x^3 + 4x^2 + x - 6$ are:
$x_1 = 1$ (Remainder = 0)
$x_2 = -3$ (Remainder = 0)
$x_3 = -2$ (Remainder = 0)
The remainder is the result of substituting the value in the equation, rounded to 10 decimal places

$7.1000x^3 - 1254x^2 - 496x + 191$

Solution
Cubic in normal form: $x^3 - 1.254x^2 - 0.496x + 0.191$
$x^3 + ax^2 + bx + c$ ----------- **[1]**
Substitute $x = t - a/3$, to eliminate the x^2 term
New equation $t^3 + pt + q = 0$
Where $p = b - a^2/3$, and $q = c + (2a^3 - 9ab) / 27$
$p = -1.020172$, $q = -0.16239726400000004$
So the new equation is:
$t^3 - 1.020172t - 0.16239726400000004$ ---------------**[2]**

The discriminant of the cubic, $D = (p/3)^3 + (q/2)^2$
Before rounding, D = –0.03273066871437038
D = –0.03273066871437
$D > 0 \rightarrow$ 2 complex roots
$D = 0 \rightarrow$ real roots, with two the same
$D < 0 \rightarrow$ Three real roots
Because $D < 0$, all the roots are real and distinct
Our reduced equation is:
$t^3 - 1.020172t - 0.16239726400000004$ **[2, repeated]**
Or in algebra, $t^3 + pt + q = 0$
There is a relationship:
$(u–v)^3 + 3uv (u – v) - (u^3 - v^3) = 0$, which indicates how we might solve the equation
If we set $t = u – v$, $p = 3uv$, or $v = p / 3u$ and $q = – (u^3 - v^3)$, that is:
$v = –1.020172 / (3u)$ ------**[4]**
$u^3 – (– 1.020172 / (3*u) - 0.16239726400000004 = 0$ ------**[5]**
So $u^3 – p^3 / (27 u^3) + q = 0$
Hence, We need to find numbers, u and v, such that:
$u^3 - v^3 = –q$, and $uv = p / 3$, giving $t = v – u$
$u^3 - v^3 = 0.16239726400000004$, -----------**[6]** and
$uv = –1.020172/3$--------- **[7]**
Using $v = –1.020172 / 3u$, we can eliminate u
$u^3 - p^3 / 27u^3 = –q$
The resulting equation is:
$27u^6 + 27qu^3 - p^3 = 0$
Normalised, this is
$u^6 + qu^3 – p^3 / 27 = 0$
$u^6 + 0.16239726400000004 * u^3 + 1.020172^3 / 27 = 0$
This is a quadratic equation in u^3, so we know how to solve it
$u^3 = – q / 2 \pm \sqrt{(q^2 /4 + p^3 / 27)}$ (5)
The square root may be negative, making the u's complex numbers, although the resulting roots might not be complex.
D = 0.02637287135468571/4 – 1.0617449369321288/27 = –0.03273066871437038
After rounding, D = – 0.03273066871437
The square root is a complex number, because $D < 0$
$u = \sqrt[3]{}$ (0.08119863200000002 $\pm \sqrt{}$(0.03273066871437) i)
We need to find the cube roots of $u^3 =$ 0.081198632 + 0.1809161925i
Convert to trigonometric form
$D = (q/2)^2 + (p/3)^3$
So $u^3 = – q/2 + \sqrt{(–D)}\, i$
As $u^3 – v^3 = – q$,
$v^3 = q / 2 + \sqrt{(–D)}\, i$
r is therefore the same for both u and v------- [A]
$r^2 = (– q/2)^2 – D$
$=(– q/2)^2 – (q/2)^2 + (p/3)^3)$
$= – (p/3)^3)$
$r = \sqrt{(– (p/3)^3)}$
Use cos to find φ, as this makes finding the angles easier (with a calculator or computer)
$r = \sqrt{((1.020172/3)^3)}$ (as it is a phasor, it can have positive values only.)
r = 0.1983025127249824 and $r^{1/3} =$ 0.5831443503398909
$\cos (\varphi) = –q/2r$

cos (φ) = 0.16239726400000004/(2*0.1983025127249824)
For v, cos(φ_v) = –0.16239726400000004/(2*0.1983025127249824)
So, cos(φ_v) = – cos(φ) [B]
Similarly, sin(φ_v) = sin(φ) [C]
acos(φ) = – q/(2*r)
φ = 1.148924921167228
x1 = u – v – a/3
u = $r^{1/3}$ * (cos(φ/3) + i sin(φ/3)
v = $r^{1/3}$*(–cos(φ/3) + i sin(φ/3) (from equations B and C above)
So, x1 = 2*r*cos(φ) – a/3 (because the sines cancel and the cosines are doubled
x1 = 2 * 0.5831443503398909 * cos(1.148924921167228) + 1.254/3
We find the other x's by adding 2π to φ, and dividing the result by 3
φ_2 = φ + 2 * π
x2 = 2 * 0.5831443503398909 * cos(2.4773700761156046) —1.254/3
φ_2 = φ + 4*π
x3 = 2 * 0.5831443503398909 * cos(4.571765178508801) —1.254/3

The roots are: $x_1 = 1.4997993055$, $x_2 = -0.5003313644$, $x_3 = 0.254532059$

The remainder is the result of substituting the value in the equation, rounded to 10 decimal places

6.7.2 AN ADDITIONAL METHOD FOR CARDON'S EQUATION

After completing theory of equations you come to this chapter, because to eliminating the second term we used that removal of a second term method in theory of equations.

1. *Solve using Cardon's method* $x^3 - 9x^2 + 108 = 0$

Solution

$x^3 - 9x^2 + 108 = 0 - - - - - - - (1)$
Remove the second term

Let $x = y + h$ and substitute in equation (1)

$(y+h)^3 - 9(y+h)^2 + 108 = 0$
$y^3 + h^3 + 3y^2h + 3yh^2 - 9y^2 - 9h^2 - 18yh + 108 = 0$
$y^3 + h^3 + y^2(3h - 9) + y(3h^2 - 18h) - 9h^2 + 108 = 0$
let us take the square term equals to zero
$3h - 9 = 0$
$h = 3$
Using synthetic division method for eliminate the second term by 3

3	1	-9	0	108
	0	3	-18	-54
	1	-6	-18	54
	0	3	-9	
	1	-3	- 27	
	0	3		
	1	0		

The transformed equation is $x^3 - 27x + 54 = 0 - - - - - - (2)$

$P = -27, q = 54$

$Substituting\ u + v = x$

$x^3 - 27x + 54 = 0$

$(u+v)^3 - 27(u+v) + 54 = 0$

$u^3 + v^3 + 3uv(u+v) - 27(u+v) + 54 = 0$

$u^3 + v^3 + (3uv - 27)(u+v) + 54 = 0$

$let\ 3uv - 27 = 0$

$uv = 27/3$

$uv = 9$

$u = 9/v\ or\ \ v = 9/u$

Then the equation is

$u^3 + v^3 + 54 = 0$

$(729/v^3) + v^3 + 54 = 0$

$729 + v^6 + 54v^3 = 0 - - - - - - - (3)$

Similarly, $729 + u^6 + 54u^3 = 0 - - - - - - - (4)$

Substituting $u^3 = v^3 = t$

$729 + t^2 + 54t = 0 - - - - - - - (5)$

$$t = \frac{-54 \pm \sqrt{2916 - 4(729)}}{2}$$

t = -27,-27

$u^3 = -27 \quad u = -3$

$v^3 = -27 \quad v = -3$

$x = u + v = -3 - 3 = -6$

diminish the equation (2) by -6

-6	1	0	-27	54
	0	-6	36	-54
	1	-6	9	0

$x^2 - 6x + 9 = 0$

$(x-3)(x-3) = 0$

$x = 3, 3$

Therefore the solutions are $x = 3,\ 3,\ -6$

2. Solve using Cardon's method $x^3 + 3x^2 + 6x + 4 = 0$

Solution

$x^3 + 3x^2 + 6x + 4 = 0 - - - - - - - (1)$

Remove the second term

Let $x = y + h$ and substitute in equation (1)

$(y+h)^3 + 3(y+h)^2 + 6(y+h) + 4 = 0$

$y^3 + h^3 + 3y^2h + 3yh^2 + 3y^2 + 3h^2 + 6yh + 6y + 6h + 4 = 0$

$y^3 + h^3 + y^2(3h+3) + y(3h^2 + 6h + 6) + 3h^2 + 6h + 4 = 0$

let us take the square term equals to zero

$3h + 3 = 0$

$h = -1$

Using synthetic division method for eliminate the second term by 3

-1	1	3	6	4
	0	-1	-2	- 4
	1	2	4	0
	0	-1	-1	
	1	1	3	
	0	-1		
	1	0		

The transformed equation is $x^3 + 3x = 0 - - - - - - (2)$

$P = 3, q = 0$

Substituting $u + v = x$

$x^3 + 3x = 0$

$(u+v)^3 + 3(u+v) = 0$

$u^3 + v^3 + 3uv(u+v) + 3(u+v) = 0$

$u^3 + v^3 + (3uv + 3)(u+v) = 0$

let $3uv + 3 = 0$

$uv = -1$

$u = -1/v$ *or* $v = -1/u$

Then the equation is $u^3 + v^3 = 0$

$\left(-1/v^3\right) + v^3 = 0$

$v^6 - 1 = 0 - - - - - - - (3)$

Similarly, $u^6 - 1 = 0 - - - - - - - (4)$

Substituting $u^3 = v^3 = t$

$t^2 - 1 = 0 - - - - - - - (5)$

$t = \pm 1$

$u^3 = \pm 1 \Rightarrow u = 1 \, or -1$

$v^3 = \pm 1 \Rightarrow v = 1 \, or - 1$

$x = u + v = 1 + 1 = 2 \, or$

$= -1 + 1 = 0 \, or$

$= -1 - 1 = -2$

$x = 2$ *is not a root of* $x^3 + 3x = 0$

$x = -2$ *is not a root of* $x^3 + 3x = 0$

$x = 0$ *is a root of* $x^3 + 3x = 0 x\left(x^2 + 3\right) = 0$

$x = 0, x^2 = -3$

$x = \pm\sqrt{3}i$

Therefore the solutions are $x = \sqrt{3}i, \; -\sqrt{3}i, \; 0$

A SIMPLE PROCEDURE FOR CARDON'S METHOD

Step 1: Let the given equation be (1)
Step 2: If the second term is present , remove the second term by diminish method in theory of equations. Let equation be (2)
Step 3: Substituting x = u + v in the transformed equation of step 2
Step 4: Equating the term to zero except the ($u^3 + v^3$ + CONSTANT), find the value of u in terms of v
Step 5: Let the equation be $u^3 + v^3$ + CONSTANT (3)
Step 6: Substituting the value of u in step 4 into the equation (3), we get the equation in terms of v (ex: $729 + v^6 + 54v^3 = 0$)
Step 7 : Similarly we get the equation in terms of u (ex: $729 + u^6 + 54u^3 = 0$)
Step 8: Substituting $u^3 = v^3 = t$, we get an equation of degree 2 in terms of t.
Step 9: Solve the equation of step 8, find two roots of t.
Step 10: Replace t by u^3, find the value of u and v.
Step 11: find x = u + v, one of the factor we can get here. Using the factor find the other two factors of the given equation.

PROBLEMS FOR PRACTICE

Find the solution using cardon's method

1. $x^3 - 6x - 4 = 0$ *answer : one real root* $x = -0.63$
2. $x^3 - 9x^2 + 108 = 0$ *answer : two real roots* $x = 3.02, - 5.43$

3. $x^3 + x^2 - 16x + 20 = 0$ *answer*: -5, $x^3 - 12x + 65 = 0$ answer: -4, $2 \pm i\sqrt{3}$

4. $x^3 - 9x + 28 = 0$ *answer*: -1, $x^3 + 3x^2 + 6x + 4 = 0$ answer: -3, $\frac{9 \pm i\sqrt{3}}{2}$

5. $x^3 - 6x^2 - 6x + 63 = 0$ *answer*: $2, 2, -5$ $x^3 + 3x^2 - 21x + 49 = 0$ answer: -7, $2 \pm i\sqrt{3}$

6. $x^3 + 6x^2 + 9x + 4 = 0$ *answer*: $-4, -1, -1$ $x^3 - 15x^2 - 33x + 847 = 0$ answer: 11,11,-7

6.8 SOLVE CUBIC EQUATIONS USING TRIGONOMETRICAL METHOD

If the three roots of a cubic equation are real, cardon's method is not applicable. In that case, the following method may be used.

We know that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0 \text{ ---------(1)}$$

Let the given equation be $x^3 + px + q = 0$ ---------(2)

put $x = r\cos\theta$ in equation (2).

Then $r^3\cos^3\theta + pr\cos\theta + q = 0$

$$\cos^3\theta + \frac{p}{r^2}\cos\theta + \frac{q}{r^3} = 0 \text{ --- ---(3)}$$

comparing (1) and (2), we get

$$\frac{p}{r^2} = -\frac{3}{4} \text{ and } \frac{q}{r^3} = -\frac{1}{4}\cos 3\theta$$

Therefore $r^2 = -\frac{4p}{3}$ and $\cos 3\theta = -\frac{4q}{r^3}$

If all the roots of the equation are real, then p is negative. Hence r^2 is real and so $r = (-\frac{4p}{3})^{\frac{1}{2}}$ is real. r is known and hence $\cos 3\theta$ is known. If α is the least value of 3θ, we get $\cos 3\theta = \cos\alpha$ or $\cos(2\pi \pm \alpha)$

Therefore $\theta = \frac{\alpha}{3}$ or $\frac{2\pi}{3} \pm \frac{\alpha}{3}$

Example 1

Solve the equation $x^3 - 6x + 4 = 0$

Solution

Here $p = -6, q = 4$

$4p^3 + 27q^2 = +ve.$

Hence all the roots of the equation are real.

If $x = r\cos\theta$, we get $r = (-\frac{4p}{3})^{\frac{1}{3}} = (\frac{24}{3})^{\frac{1}{2}} = 2\sqrt{2}$

and $\cos 3\theta = \frac{-4q}{r^3} = \frac{-4(4)}{(2\sqrt{2})^3} = -\frac{1}{\sqrt{2}}$

Hence the least value of 3θ is $\frac{3\pi}{4}$.

$\text{Cos } 3\theta = \cos\frac{3\pi}{4} \text{ or} \cos\ (2\pi \pm \frac{3\pi}{4})$

Therefore $\theta = \frac{\alpha}{3}$ or $\frac{2\pi}{3} \pm \frac{\alpha}{3}$

Hence the root of the equation are

$2\sqrt{2}\cos\frac{\pi}{4}, 2\sqrt{2}\cos(\frac{2\pi}{3} \pm \frac{\pi}{4})$

$2, 2\sqrt{2}\cos\frac{5\pi}{12}, 2\sqrt{2}\cos(\frac{11\pi}{12})$

$2, 2\sqrt{2}\cos 75^\circ, 2\sqrt{2}\cos 15$

$2 \ and -1 \pm\sqrt{3}$

otherwise , 2 is a root of the equation dividing the equation by $x-2$, we get $x^2 + 2x - 2 = 0$ whose roots are $-1 \pm\sqrt{3}$

Example 2

Solve the equation $x^3 - 3x + 1 = 0$

Solution

Here p = -3, q = 1

Therefore $r = (-\frac{4p}{3})^{\frac{1}{3}}$ = $2\cos 3\theta$ = $\frac{-4q}{r^3}$ = $-\frac{1}{2}$

$\text{Cos } 3\theta$ = $\cos\frac{2\pi}{3}$

Hence

$\theta = \frac{2\pi}{9} or \frac{2\pi}{3} \pm \frac{2\pi}{9}$

$\theta = \frac{2\pi}{9} or \frac{4\pi}{9} or \frac{8\pi}{9}$

Hence the roots of the equation are

$2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9} and 2\cos\frac{8\pi}{9}$

REFERENCES

1. Curtis F.Gerald, Patrick O. Wheatley- Applied Numerical Analysis- Addisen Wesley Longman Publishing Co(1989).
2. Manickavasagam Pillai. T.K. A Text Book of Algebra – S.Viswanathan (Printers and Publishers), Pvt Ltd. 2004.
3. Kanna .M.L., ALGEBRA- Jai Prakash Nath & Co., - 1982.
4. Srimanta Pal, Subodh C. Bhunia., Engineering Mathemtics- OXFORD University Press-2015.
5. Vittal. P.R, Malini.V., Algebra, Analytical Geometry and Trigonometry- Margham Publication – 1998.
6. http://artofproblemsolving.com/wiki/index.php/2006_AIME_I_Problems
7. www.cengage.com/resource_uploads/downloads/084006862X_397763.pdf
8. https://www.scribd.com/document/361109599/Theory-of-Equations
9. http://tutorial.math.lamar.edu/Classes/CalcI/NewtonsMethod.aspx
10. http://missbrownsmathclass.weebly.com/uploads/3/2/1/6/32168659/3.8_notes.pdf
11. http://researchhubs.com/post/maths/fundamentals/newtons-method.html
12. https://en.mimi.hu/mathematics/iteration.html
13. http://mnmathleague.org/wp-content/uploads/2012/09/Partial-Fraction-Decomposition.pdf
14. http://tutorial.math.lamar.edu/Classes/CalcII/PartialFractions.aspx
15. http://tutorial.math.lamar.edu/classes/alg/PartialFractions.aspx
16. www2.trinity.unimelb.edu.au/~rbroekst/MathX/Quartic%20Formula.pdf

MODEL QUESTION PAPER- I

Degree: B.Sc.,(Mathematics)

Time: 3hrs Max Marks: 100

CLASSICAL ALGEBRA

I. ANSWER THE FOLLOWING (10X2= 20)

1. Use the binomial theorem to find the 7^{th} power of 11
2. Find the co – efficient of x^n in the expansion of $\frac{1+2x-3x^2}{e^x}$
3. Find the coefficient of x^n in the expansion of $e^{a + bx}$
4. Solve the equation $x^3 - 12x^2 + 39x - 28 = 0$ whose roots are in A.P.
5. Define a reciprocal equation
6. Prove that the equation $x^3 + 2x + 3 = 0$ has one negative root and two imaginary root.
7. Define strum's functions
8. Write the step by step procedure for Horner's method.
9. Find the equation with 3 unknowns $4x^4 + 4x^3 - 7x^2 - 4x - 12 = 0$ using biquadratic form no need to solve.
10. What is the use of Cardon's method?

II. ANSWER THE FOLLOWING (5X 16 = 80)

11. Sum to infinity the series $\frac{7}{72}+\frac{7}{72}.\frac{28}{96}+\frac{7}{72}.\frac{28}{96}.\frac{49}{120}+............\infty$ (Or)

12. Sum to infinity the series $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+............\infty$

13. If α be a real root of the cubic equation $x^3+px^2+qx+r=0$, of which the co efficients are real, show that the other two roots of the equation are real, if $p^2 \geq 4q + 2p\alpha + 3\alpha^2$ (or)

14. Show that the sum of the eleventh powers of the roots of $x^7 + 5x^4 + 1 = 0$ is zero.

15. Find the roots of the equation $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1 = 0$(or)

16. Show that the equation $3x^4 - 8x^3 - 6x^2 + 24x - 7 = 0$ has one +ve one -ve and two imaginary roots.

17. Find the number and position of real roots of the equation $x^3 - 3x + 6 = 0$ using strum's function.(or)

18. The equation $x^3 - 3x + 1 = 0$ has a root between 1 and 2. Calculate it to three places of decimals by Horner's method.(or)

19. Solve the equation $x^3 - 6x - 9 = 0$ using cardon's method(or)

20. Solve the equation $x^4 + 4x^3 - 7x^2 - 4x - 12 = 0$ using biquadratic equation.

MODEL QUESTION PAPER-II

Degree: B.Sc.,(Mathematics)

Time: 3hrs Sub : Classical Algebra Max Marks: 100

I. Answer The Following (10X2= 20)

1. Find the sum of the coefficients of the first (r + 1) terms in the expansion of $(1-x)^{-3}$
2. Using the binomial theorem for rational index, Find the value of $\frac{1}{(9998)^{\frac{1}{4}}}$.
3. Write the Remainder theorem.
4. Form a cubic equation which shall have roots 1, 3- $\sqrt{2}$
5. Change the equation $2x^4 - 3x^3 + 3x^2 - x + 2 = 0$ into another the coefficient of whose highest term will be unity.
6. Find the roots of the equation $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$
7. How to find the number of negative roots in strum's function
8. Solve the equation $x^4 + 2x^3 - x - 2 = 0$. The integral roots must be found among the values $\pm 1, \pm 2$.
9. Write the general procedure for cardon's method
10. Solve the equation $x^3 - 6x - 9 = 0$

II. Answer the following (5X 16 = 80)

11. Expand $\frac{1}{(1-2x)(1+3x)}$ in a series of ascending powers of x, and find when the expansion is valid. (or)

12. Sum to infinity the series $\frac{1^2}{1!} + \frac{1^2+2^2}{2!} + \frac{1^2+2^2+3^2}{3!} + \ldots\ldots + \frac{1^2+2^2+\ldots+n^2}{n!} + \ldots\ldots\infty$

13. If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two, prove that $p^3 + 8r = 4pq$ (or)

14. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$ form the equation whose roots are $\beta + \gamma - 2\alpha$, $\gamma + \alpha - 2\beta$, $\alpha + \beta - 2\gamma$

15. Solve the equation $x^4 + 20x^3 - 143x^2 + 430x + 462 = 0$ by removing its second term. (or)

16. Find the nature of the roots of the equation $4x^3 - 21x^2 + 18x + 20 = 0$ using Rolle's theorem.

17. Find the number and position of real roots of the equation $x^4 - 14x^2 + 16x + 9 = 0$ using strum's function.(or)

18. Solve The equation $x^3 – 2x^2 - 3x - 4 = 0$ Calculate it to three places of decimals by Horner's method.

19. Solve the equation $x^3 - 9x^2 + 108 = 0$ using Cardon's method (or)

20. Solve the equation $x^4 - 4x^3 - 10x^2 + 64x + 40 = 0$ using biquadratic equation.

MODEL QUESTION PAPER-III

Degree: B.Sc.,(Mathematics) Time: 3hrs

Classical algebra Max marks :100

I. Answer the following (10X2= 20)

11. Use the binomial theorem to find the sum to infinity of the series
$$1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+........$$
12. Show that $\sqrt{x^2+16}-\sqrt{x^2+9}=\frac{7}{2x}$ nearly for sufficiently large values of x.
13. Form a cubic equation which shall have roots 1, i .
14. Solve the $81x^3 - 18x^2 + 36x + 8 = 0$ whose roots are in H.P.
15. Solve the equation $x^4+3x^3-3x-1=0$
16. Find the quotient and remainder when $3x^3 + 8x^2 + 8x + 12 = 0$ is divided by x-4.
17. Remove the second term $x^3 - 6x^2 + 10x – 3 = 0$
18. Find the rational root of $2x^3 – x^2 – x - 3 = 0$ and hence complete the solution of the equation.
19. Solve the equation $x^3 - 3x + 1 = 0$ using trignometrical method.
20. How to do the problem with second term in Cardon's method?

II. Answer The Following (5x 16 = 80)

11. Sum to infinity the series $\frac{1}{24}-\frac{1.3}{24.32}+\frac{1.3.5}{24.32.40}-............\infty$ (or)

12. Sum to infinity the series $\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+............\infty = 3\log 2 - 1.$

13. find the condition that the roots of the equation a $x^3 + 3bx^2 + 3cx + d = 0$ may be in Geometric progression. Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in G.P (or)

14. If a, b, c are the roots of $x^3 + px^2 + qx + r = 0$ form the equation whose roots are b + c - 2a, c + a - 2b, a + b - 2c.

15. Increase the roots by 7, $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$ (or)

16. Discuss the reality of roots $x^4 + 4x^3 - 2x^2 - 12x + a = 0$ for all values of a using Roll's theorem.

17. Find the number and position of real roots of the equation $x^4 + 4x^3 - 4x - 13 = 0$ using strum's function.(or)

18. The equation $x^3 + 6x - 2 = 0$ has a root between 0 and 1. Calculate it to three places of decimals by Horner's method.(or)

19. Solve the equation $x^3 - 3x + 1 = 0$ using Cardon's method

(or)

20. Solve the equation $2x^4 + 6x^3 - 3x^2 + 2 = 0$ using biquadratic(Ferrari) method .

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